AS

Further Mathematics

Paper 1

Mark scheme

Specimen

Version 1.1

Mark schemes are prepared by the Lead Assessment Writer and considered, together with the relevant questions, by a panel of subject teachers. This mark scheme includes any amendments made at the standardisation events which all associates participate in and is the scheme which was used by them in this examination. The standardisation process ensures that the mark scheme covers the students' responses to questions and that every associate understands and applies it in the same correct way. As preparation for standardisation each associate analyses a number of students' scripts. Alternative answers not already covered by the mark scheme are discussed and legislated for. If, after the standardisation process, associates encounter unusual answers which have not been raised they are required to refer these to the Lead Assessment Writer.

It must be stressed that a mark scheme is a working document, in many cases further developed and expanded on the basis of students' reactions to a particular paper. Assumptions about future mark schemes on the basis of one year's document should be avoided; whilst the guiding principles of assessment remain constant, details will change, depending on the content of a particular examination paper.

Further copies of this mark scheme are available from aqa.org.uk

Mark scheme instructions to examiners

General

The mark scheme for each question shows:

- the marks available for each part of the question
- the total marks available for the question
- marking instructions that indicate when marks should be awarded or withheld including the principle on which each mark is awarded. Information is included to help the examiner make his or her judgement and to delineate what is creditworthy from that not worthy of credit
- a typical solution. This response is one we expect to see frequently. However credit must be given on the basis of the marking instructions.

If a student uses a method which is not explicitly covered by the marking instructions the same principles of marking should be applied. Credit should be given to any valid methods. Examiners should seek advice from their senior examiner if in any doubt.

Key to mark types

M	mark is for method
dM	mark is dependent on one or more M marks and is for method
R	mark is for reasoning
A	mark is dependent on M or m marks and is for accuracy
B	mark is independent of M or m marks and is for method and accuracy
E	mark is for explanation
F	follow through from previous incorrect result

Key to mark scheme abbreviations

CAO	correct answer only
CSO	correct solution only
ft	follow through from previous incorrect result
‘their’	Indicates that credit can be given from previous incorrect result
AWFW	anything which falls within
AWRT	anything which rounds to
ACF	any correct form
AG	answer given
SC	special case
OE	or equivalent
NMS	no method shown
PI	possibly implied
SCA	substantially correct approach
sf	significant figure(s)
dp	decimal place(s)

Examiners should consistently apply the following general marking principles

No Method Shown

Where the question specifically requires a particular method to be used, we must usually see evidence of use of this method for any marks to be awarded.

Where the answer can be reasonably obtained without showing working and it is very unlikely that the correct answer can be obtained by using an incorrect method, we must award **full marks**. However, the obvious penalty to candidates showing no working is that incorrect answers, however close, earn **no marks**.

Where a question asks the candidate to state or write down a result, no method need be shown for full marks.

Where the permitted calculator has functions which reasonably allow the solution of the question directly, the correct answer without working earns **full marks**, unless it is given to less than the degree of accuracy accepted in the mark scheme, when it gains **no marks**.

Otherwise we require evidence of a correct method for any marks to be awarded.

Diagrams

Diagrams that have working on them should be treated like normal responses. If a diagram has been written on but the correct response is within the answer space, the work within the answer space should be marked. Working on diagrams that contradicts work within the answer space is not to be considered as choice but as working, and is not, therefore, penalised.

Work erased or crossed out

Erased or crossed out work that is still legible and has not been replaced should be marked. Erased or crossed out work that has been replaced can be ignored.

Choice

When a choice of answers and/or methods is given and the student has not clearly indicated which answer they want to be marked, only the last complete attempt should be awarded marks.

Q	Marking Instructions	AO	Mark	Typical Solution
1	Circles correct answer	AO1.1b	B1	$y = 0$
	Total		**1**	
2	Circles correct answer	AO1.1b	B1	13
	Total		**1**	
3	Circles correct answer	AO1.1b	B1	$y = \pm\frac{1}{\sqrt{3}}x$
	Total		**1**	
4(a)	Finds $k = 2$	AO1.1b	B1	$k - 2 = 0 \Rightarrow k = 2$
4(b)	States correct transformation	AO1.2	B1	Reflection in the y-axis
4(c)(i)	Finds product **BA** Allow one slip	AO1.1a	M1	$\mathbf{BA} = \begin{bmatrix} -1 & -2 \\ 1 & k \end{bmatrix}$
	Obtains inverse FT 'their' **BA** provided M1 awarded	AO1.1b	A1F	$(\mathbf{BA})^{-1} = \frac{1}{-k-(-2)} \begin{bmatrix} k & 2 \\ -1 & -1 \end{bmatrix}$
	Finds $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$	AO1.1b	B1	$\mathbf{A}^{-1} = \frac{1}{k-2} \begin{bmatrix} k & -2 \\ -1 & 1 \end{bmatrix}$
	Obtains correct $\mathbf{A}^{-1}\mathbf{B}^{-1}$ and shows that $(k-2)\times(-1) = 2-k = -k-(-2)$ thus completing verification Must clearly show $\mathbf{A}^{-1} \times \mathbf{B}^{-1}$ method for this mark – disallow if answer simply stated	AO2.1	R1	$\mathbf{B}^{-1} = \frac{1}{-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\mathbf{A}^{-1}\mathbf{B}^{-1} = \frac{1}{(k-2)\times(-1)}$ $\begin{bmatrix} k+0 & (-2\times-1)+0 \\ (-1\times1)+0 & 0+(-1\times1) \end{bmatrix}$ That is the same as $(\mathbf{BA})^{-1}$ $(\mathbf{BA})^{-1} = \frac{1}{2-k}\begin{bmatrix} k & 2 \\ -1 & -1 \end{bmatrix} = \mathbf{A}^{-1}\mathbf{B}^{-1}$

Q	Marking Instructions	AO	Mark	Typical Solution
4(c)(ii)	Uses equation for identity from definition	AO3.1a	M1	We require to demonstrate that: $(\mathbf{NM}) \times \{\mathbf{M}^{-1}\mathbf{N}^{-1}\} = \boldsymbol{I}$
	Comences argument by manipulating the matrix products within the equation with clear pairing	AO2.1	R1	$(\mathbf{NM}) \times \mathbf{M}^{-1}\mathbf{N}^{-1} = \mathbf{N}(\mathbf{M} \times \mathbf{M}^{-1})\mathbf{N}^{-1}$ $= \mathbf{N}\,\boldsymbol{I}\,\mathbf{N}^{-1}$ $= \mathbf{NN}^{-1}$
	Clearly demonstrates that $\mathbf{M} \times \mathbf{M}^{-1} = \boldsymbol{I}$ used	AO2.4	B1	$= \boldsymbol{I}$
	Completes the argument using rigorous reasoning with definition of matrix inverse and associativity mentioned Must see all working with correct pairing of each matrix with inverse	AO2.1	R1	Using definition of matrix inverse and associativity of matrix multiplication Hence true for all non-singular matrices **N** and **M**
	Total		**10**	

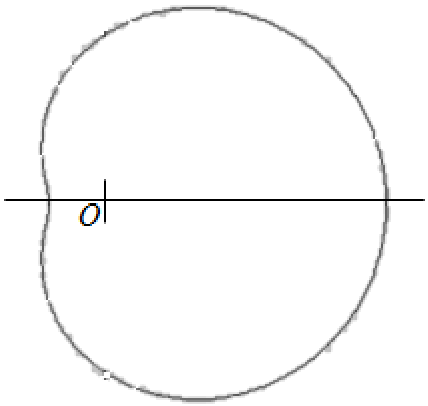
Q	Marking Instructions	AO	Mark	Typical Solution
5	Obtains y^2	AO1.1b	B1	$y^2 = 9 + 6\sqrt{x} + x$
	Integrates 'their' y^2 within the integral to find the volume of revolution with at least two terms correct (condone missing π)	AO1.1a	M1	Volume $= \pi\int_1^4 (9 + 6\sqrt{x} + x)\,dx$ $= \pi\left[9x + 4x^{\frac{3}{2}} + \frac{x^2}{2}\right]_1^4$
	Obtains all terms correctly FT 'their' y^2, provided M1 awarded	AO1.1b	A1F	Substituting limits
	Substitutes correct limits into 'their' volume expression FT provided previous M1 awarded	AO1.1a	M1	$\left[9\times4+4\times4^{\frac{3}{2}}+\frac{4^2}{2}\right]-\left[9\times1+4\times1^{\frac{3}{2}}+\frac{1^2}{2}\right]$ $=76-13\frac{1}{2}=62\frac{1}{2}=\frac{125}{2}$
	Completes a fully correct argument to obtain correct expression with $\pi\int y^2 dx$ found at some earlier stage in the working **AG**	AO2.1	A1	Hence volume = $\frac{125\pi}{2}$ **AG**
	Total		**5**	

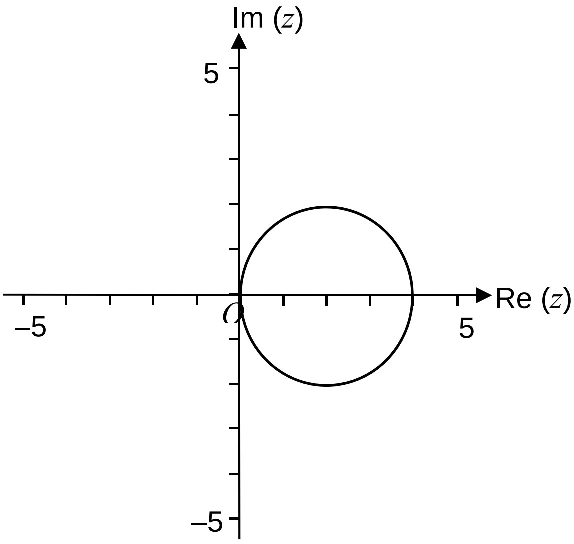
Q	Marking Instructions	AO	Mark	Typical Solution
6(a)	Uses definitions of sinh x and cosh x to obtain expression for tanh x	AO1.2	B1	$\sinh x = \dfrac{e^x - e^{-x}}{2}$
	Multiplies by e^x	AO1.1a	M1	$\cosh x = \dfrac{e^x - e^{-x}}{2}$
	Obtains e^{2x}	AO1.1b	A1	$\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$
	Completes a fully correct argument by demonstrating result by taking logs This mark is available only if all previous marks have been awarded	AO2.1	R1	Multiplying numerator and denominator by e^x $t = \dfrac{e^{2x} - 1}{e^{2x} + 1}$ $te^{2x} + t = e^{2x} - 1$ [or multiplies by e^x in $te^x + te^{-x} = e^x - e^{-x}$] $1 + t = e^{2x}(1 - t)$ $e^{2x} = \dfrac{1+t}{1-t}$ $2x = \ln\dfrac{1+t}{1-t}$ hence $x = \dfrac{1}{2}\ln\dfrac{1+t}{1-t}$

Q	Marking Instructions	AO	Mark	Typical Solution
6(b)(i)	Expresses cosh $3x$ and cosh x in exponential form Seen anywhere in solution	AO1.2	B1	To be proven: $\left(\frac{e^{x}+e^{-x}}{2}\right)^3 =$ $\frac{1}{4}\left(\frac{e^{3x}+e^{-3x}}{2}\right)+\frac{3}{4}\left(\frac{e^{x}+e^{-x}}{2}\right)$ LHS $\left(\frac{e^{x}+e^{-x}}{2}\right)^3 =$ $\frac{1}{8}(e^{3x}+3e^{2x}.e^{-x}+3e^{x}.e^{-2x}+e^{-3x}) =$ $\frac{1}{4}\left(\frac{(e^{3x}+e^{-3x})}{2}\right)+3\frac{\left(e^{x}+e^{-x}\right)}{2} =$ $\frac{1}{4}\cosh 3x+\frac{3}{4}\cosh x = \text{RHS}$ From the definition of cosh x
	Expands LHS FT 'their' LHS provided first M1 awarded Allow one slip	AO1.1a	M1	
	Simplifies and collects terms FT 'their' expression Allow one slip	AO1.1a	M1	
	Completes fully correct proof to reach the required result This mark is available only if all previous marks have been awarded	AO2.1	R1	
6(b)(ii)	Substitutes for cosh $3x$ in equation from part **(b)(i)** Allow one slip	A03.1a	M1	$\left(\cosh x\right)^3 = \frac{1}{4}\times 13\cosh x+\frac{3}{4}\cosh x$ $\left(\cosh x\right)^3 - 4\cosh x = 0$ $\cosh x\left[\left(\cosh x\right)^2 - 4\right] = 0$ Solutions are $\cosh x = 0, -2, 2$ solutions 0 and -2 are not possible since range of $\cosh x \geq 1$ $\cosh x = 2 \Rightarrow x = \ln\left(2+\sqrt{3}\right)$
	Obtains equation in cosh x and solves it Allow one slip	AO1.1a	M1	
	Eliminates 0 and –2 with reason	AO2.4	E1	
	States correct solution in exact log form	AO1.1b	A1	
	Total		**12**	

Q	Marking Instructions	AO	Mark	Typical Solution
7(a)	Models light beams as straight lines and forms vector equations for straight lines using a suitable origin	AO3.3	M1	Modelling beams of light as straight lines taking the origin as point *A*: $\mathbf{r}_A = \begin{pmatrix}0\\0\\3\end{pmatrix} + \lambda\left(\begin{pmatrix}30\\125\\23\end{pmatrix} - \begin{pmatrix}0\\0\\3\end{pmatrix}\right)$
	Forms correct vector equation for a line. Allow one slip	AO1.1b	A1	$= \begin{pmatrix}0\\0\\3\end{pmatrix} + \lambda\begin{pmatrix}30\\125\\20\end{pmatrix}$
	Forms correct vector equation for second line. Allow one slip	AO1.1b	A1	$\mathbf{r}_B = \begin{pmatrix}8\\10\\1\end{pmatrix} + \mu\left(\begin{pmatrix}28\\140\\29\end{pmatrix} - \begin{pmatrix}8\\10\\1\end{pmatrix}\right)$
	Forms equations for two components using 'their' model FT 'their' lines	AO3.4	M1	$= \begin{pmatrix}8\\10\\1\end{pmatrix} + \mu\begin{pmatrix}20\\130\\28\end{pmatrix}$ $30\lambda = 8 + 20\mu$ $125\lambda = 10 + 130\mu$
	Solves 'their' equations correctly FT 'their' lines	AO1.1b	A1F	$\lambda = \frac{3}{5}$ and $\mu = \frac{1}{2}$
	Checks with third component and concludes that the beams of light intersect This mark is available only if all previous marks have been awarded	AO2.1	R1	$3 + \frac{3}{5} \times 20 = 15$ $1 + \frac{1}{2} \times 28 = 15$ $\therefore$ Intersect

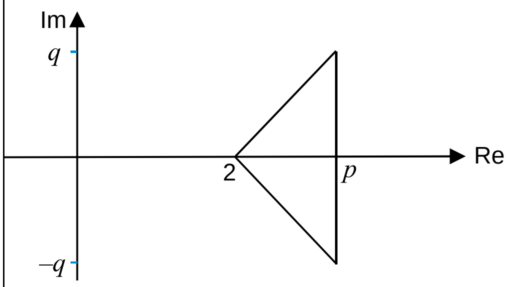
Q	Marking Instructions	AO	Mark	Typical Solution
7(b)	Evaluates scalar product for ‘their’ direction vectors. (PI)	AO3.1a	M1	$\begin{pmatrix}30\\125\\20\end{pmatrix}\bullet\begin{pmatrix}20\\130\\28\end{pmatrix}=17410$
	Sets up equation to find angle. (PI) FT only if previous M1 awarded	AO1.1a	M1	$\cos\theta=\frac{17410}{\sqrt{30^2+125^2+20^2}\times\sqrt{20^2+130^2+28^2}}$ $\cos\theta=\frac{17410}{\sqrt{16925}\times\sqrt{18084}}=0.9951$
	Obtains correct angle.	AO1.1b	A1	$\theta=5.6^\circ$
7(c)	States appropriate refinement.	AO3.5c	E1	Take account of the width of the beams.
	Total		**10**	

Q	Marking Instructions	AO	Mark	Typical Solution
8(a)(i)	States max value for r	AO1.1b	B1	Maximum value of $r = 5$
	States min value for r	AO1.1b	B1	Minimum value of $r = 1$
(ii)	Draws simple closed curve enclosing pole	AO1.1a	M1	
	Draws correct shape with dimple (not cusp) when $\theta = \pi$	AO1.1b	A1	
(b)	Equates $3 + 2\cos\theta = 8\cos^2\theta$	AO1.1a	M1	$3 + 2\cos\theta = 8\cos^2\theta$
	Solves 'their' quadratic equation FT 'their' equation only if M1 has been awarded	AO1.1a	M1	$8\cos^2\theta - 2\cos\theta - 3 = 0$ $(4\cos\theta - 3)(2\cos\theta + 1) = 0$
	Obtains 2 values for θ for each value of $\cos\theta$ FT 'their' equation only if both M1 marks have been awarded	AO1.1b	A1F	$\cos\theta = \frac{3}{4}, \quad \cos\theta = -\frac{1}{2}$
	Substitutes 'their' $\cos\theta$ into a polar equation to find a value of r FT 'their' $\cos\theta$ only if both M1 marks have been awarded	AO1.1a	M1	$\theta = 0.723$ or $\frac{2\pi}{3}$ $\theta = 5.56$ or $\frac{4\pi}{3}$
	Obtains both values of r correct for 'their' $\cos\theta$ values FT 'their' $\cos\theta$ only if both M1 marks have been awarded	AO1.1b	A1F	$\cos\theta = \frac{3}{4} \Rightarrow r = \frac{9}{2}$ $\cos\theta = -\frac{1}{2} \Rightarrow r = 2$
	Deduces that four values for θ exist and expresses points in required form	AO2.2a	R1	Intersection points $\left[\frac{9}{2}, 0.723\right]$, $\left[\frac{9}{2}, 5.56\right], \left[2, \frac{2\pi}{3}\right], \left[2, \frac{4\pi}{3}\right]$
	Total		**10**	

Q	Marking Instructions	AO	Mark	Typical Solution
9(a)	Draws 'circle' with centre $2 + 0\mathrm{i}$ Ignore other features	AO1.1a	M1	
	Draws circle passing through (0, 0), (4, 0), close to (2, 2) and $(2, -2)$ with Imaginary axis tangential	AO1.1b	A1	
(b)	Uses mod/arg forms	AO3.1a	M1	$z-2=2\left[\cos\left(-\frac{\pi}{3}\right)+\mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right]$
	Substitutes exact values for cos and sin Allow one slip	AO1.1a	M1	$=2\left[\frac{1}{2}+\mathrm{i}\left(-\frac{\sqrt{3}}{2}\right)\right]$
	Obtains result in exact form	AO1.1b	A1	$z=3-\sqrt{3}\,\mathrm{i}$
	Total		**5**	

Q	Marking Instructions	AO	Mark	Typical Solution
10(a)	Splits up the sum into separate sums $\sum ar^2 + \sum br + (\sum c)$ PI	AO3.1a	M1	$\sum_{r=1}^{n}(r+1)(r+2) = \sum_{r=1}^{n}(r^2+3r+2)$ $= \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} 3r + \sum_{r=1}^{n} 2$
	Substitutes for the two sums $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r$ Allow one slip	AO1.1a	M1	$S = \frac{n}{6}(n+1)(2n+1) + 3\frac{n}{2}(n+1) + \sum_{r=1}^{n} 2$
	States or uses $\sum_{r=1}^{n} 1 = n$ PI	AO1.2	B1	$= \frac{n}{6}(n+1)(2n+1) + 3\frac{n}{2}(n+1) + 2n$ Now $6 + 3\sum_{r=1}^{n}(r+1)(r+2)$
	Factorises out $(n+1)$ Allow one slip	AO1.1a	M1	$= 6 + \frac{n}{2}(n+1)(2n+1) + 9\frac{n}{2}(n+1) + 6n$
	Simplifies $(n+1)\{\frac{n}{2}(2n+1) + \frac{9n}{2} + 6\}$ to find second linear factor from 'their' quadratic FT 'their' quadratic provided all M1 marks have been awarded Allow one slip	AO1.1a	M1	$= \frac{n}{2}(n+1)(2n+1) + \frac{9n}{2}(n+1) + 6(n+1)$ $= (n+1)\{\frac{n}{2}(2n+1) + \frac{9n}{2} + 6\}$ $= (n+1)(n^2+5n+6)$
	Completes a rigorous argument to show the required result To obtain this mark factorising must be clearly seen and all previous marks obtained	AO2.1	R1	$= (n+1)(n+2)(n+3)$

Q	Marking Instructions	AO	Mark	Typical Solution
10(b)	Chooses a multiple of 4 for n and obtains a correct numerical value/expression	AO2.4	E1	When $n = 4$, $6+3\sum_{r=1}^{n}(r+1)(r+2) = (5)(6)(7)$
	Clear argument with concluding statement	AO2.3	E1	= 210 which is not a multiple of 12 so Alex's statement is false.
	Total		**8**	

Q	Marking Instructions	AO	Mark	Typical Solution
11	Writes β and γ in the form $p \pm q\text{i}$ (seen anywhere in the solution)	AO2.5	B1	Real coefficients $\Rightarrow$ $\beta = p + q\text{i}$ and $\gamma = p - q\text{i}$
	Uses "sum of the roots = –b/a" together with a conjugate pair to determine the real part (p) of β and γ	AO3.1a	M1	$\alpha + \beta + \gamma = 8$ $\Rightarrow 2 + p + q\text{i} + p - q\text{i} = 8$ $\Rightarrow 2 + 2p = 8$ $\Rightarrow p = 3$
	Uses '(their p)' –2 and the area of the triangle on an Argand diagram to determine the imaginary parts of β and γ	AO3.1a	M1	$(p-2)q = 8$ $\Rightarrow q = 8$ 
	Uses a correct method to find the value of c or d using 'their' values of $p \pm q\text{i}$	AO1.1a	M1	$\beta = 3 + 8\text{i}$ and $\gamma = 3 - 8\text{i}$
	Obtains correct values for c and d. CAO	AO1.1b	A1	$d = -\alpha\beta\gamma = -146$ $c = \sum \alpha\beta = 85$
	Total		**5**	

Q	Marking Instructions	AO	Mark	Typical Solution
12(a)(i)	Eliminates y	AO1.1a	M1	$k=\dfrac{5x^2-12x+12}{x^2+4x-4}$
	Obtains a quadratic equation in the form $Ax^2+Bx+C=0$, PI by later work	AO3.1a	M1	$k\left(x^2+4x-4\right)=5x^2-12x+12$ $(k-5)x^2+4(k+3)x-4(k+3)=0$ (A)
	Obtains b^2-4ac in terms of k for 'their' quadratic FT 'their' quadratic provided first M1 awarded	AO1.1b	A1F	$y=k$ intersects C_1 so roots of (A) are real $b^2-4ac=$
	Obtains inequality, including ≥ 0, where k is the only unknown for 'their' discriminant FT 'their' discriminant provided both M1 marks have been awarded	AO1.1a	M1	$[4(k+3)]^2-4(k-5)(-4(k+3))$ $16\left(k+3\right)^2+16\left(k-5\right)\left(k+3\right)\geq 0$ $16\left(k+3\right)\left(k+3+k-5\right)\geq 0$
	Completes a rigorous argument to show that $\left(k+3\right)\left(k-1\right)\geq 0$ This mark is available only if all previous marks have been awarded	AO2.1	R1	$\Rightarrow\ \left(k+3\right)\left(2k-2\right)\geq 0$ $\Rightarrow\ \left(k+3\right)\left(k-1\right)\geq 0$
12(a)(ii)	Obtains critical values	AO1.1b	B1	Critical values are -3 and 1
	Deduces that $k=-3$ for maximum	AO2.2a	R1	$k\leq -3$ (or $k\geq 1$) satisfy inequality
	Substitutes for k into 'their' quadratic from **(a)(i)** FT 'their' quadratic only if first M1 awarded in **(a)(i)**	AO1.1a	M1	For max pt, $k=-3$ Sub $k=-3$ in (A) gives $-8x^2=0$
	States coordinates of max pt NMS 0/4 Must be using **(a)(i)**	AO1.1b	A1 CAO	Max pt of C_1 is $(0,-3)$

Q	Marking Instructions	AO	Mark	Typical Solution
12(b)	Uses discriminant to determine solution	AO2.4	E1	$(-12)^2 - 4(5)(12) < 0$ $k \neq 0$
	Deduces no vertical asymptotes with clear reasoning with reference to denominator	AO2.2a	R1	Denominator, $5x^2 - 12x + 12$ of $\dfrac{1}{\mathrm{f}(x)}$ is never 0 so C_2 has no vertical asymptotes.
12 (c)	Obtains $y = 1$	AO3.2a	B1	$y = 1$
	Total		**12**	
	TOTAL		**80**	