Teacher Support Materials 2009

Maths GCE

Paper Reference MFP1

Question 1

1 The equation

$$2x^2 + x - 8 = 0$$

has roots α and β.

(a) Write down the values of $\alpha + \beta$ and $\alpha\beta$. *(2 marks)*

(b) Find the value of $\alpha^2 + \beta^2$. *(2 marks)*

(c) Find a quadratic equation which has roots $4\alpha^2$ and $4\beta^2$. Give your answer in the form $x^2 + px + q = 0$, where p and q are integers. *(3 marks)*

Student Response

1 $2x^2 + x - 8 = 0$

$a = 2 \quad b = 1 \quad c = -8.$

(a) $\alpha + \beta = -\dfrac{b}{a} \rightarrow \dfrac{-1}{2} = -\frac{1}{2}.$ ✓

$\alpha\beta = \dfrac{c}{a} = -8 \times 2 = -16$ ✗

(b) $\alpha^2 + \beta^2.$

$(\alpha^2 + \beta^2) = (\alpha+\beta)^2.$ ✗

$(\alpha^2 + \beta^2) = \alpha^2 + \beta^2 + 2\alpha\beta.$ ✗

$(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta.$

$(-0.5)^2 = \alpha^2 + \beta^2 + 2 \times -16.$

$0.25 = \alpha^2 + \beta^2 + -32.$

$32.25 = \alpha^2 + \beta^2.$ ✗

(c) $4\alpha^2, 4\beta^2.$

$4(\alpha^2 + \beta^2) \rightarrow 4(32.25) = 129$ ✓

$4\alpha^2, 4\beta^2$

~~$4(\alpha^2 + \beta$~~

$4^2(\alpha^2\beta^2) \rightarrow \cancel{16}\,(4(\alpha\beta)^2 \rightarrow 4(-16)^2$

$\rightarrow 4(256).$

$= 1024.$ ✗

$\rightarrow x^2 - 129x + 1024 = 0.$

Commentary

Most candidates, as shown in this example, found the sum and product of the roots correctly. Unfortunately, as shown in this example, the algebraic skills of some of the candidate was not sufficient to find $(\alpha^2 + \beta^2)$. The other common error, also shown here, was finding the product of the new roots which was $4\alpha^2 \times 4\beta^2$ or $16\alpha^2\beta^2$; many used its value as $4(\alpha\beta)^2$

Mark scheme

1(a)	$\alpha+\beta=-\frac{1}{2}$, $\alpha\beta=-4$	B1B1	2	
(b)	$\alpha^2+\beta^2=(-\frac{1}{2})^2-2(-4)=8\frac{1}{4}$	M1A1F	2	M1 for substituting in correct formula; ft wrong answer(s) in (a)
(c)	Sum of roots = $4(8\frac{1}{4})=33$	B1F		ft wrong answer in (b)
	Product = $16(\alpha\beta)^2=256$	B1F		ft wrong answer in (a)
	Equation is $x^2-33x+256=0$	B1F	3	ft wrong sum and/or product; allow '$p=-33$, $q=256$'; condone omission of '= 0'
	Total		**7**	

Question 2

2 A curve has equation

$$y = x^2 - 6x + 5$$

The points A and B on the curve have x-coordinates 2 and $2 + h$ respectively.

(a) Find, in terms of h, the gradient of the line AB, giving your answer in its simplest form. *(5 marks)*

(b) Explain how the result of part (a) can be used to find the gradient of the curve at A. State the value of this gradient. *(3 marks)*

Student response

2a) $y = x^2 - 6x + 5$

$x_A = 2$

$x_B = 2+h$

$y = 4 - 12 + 5$

$= -3$

$(2, -3)$

$y = 4 + 4h + h^2 - 12 - 6h + 5$

$= h^2 - 2h - 3$

$(2+h, h^2 - 2h - 3)$

gradient AB

$= \dfrac{y_B - y_A}{x_B - x_A}$

$= \dfrac{h^2 - 2h - 3 - -3}{2 + h - 2}$

$= \dfrac{h^2 - 2h}{h} = h - 2$

2b) ~~[illegible]~~ $\dfrac{dy}{dx} = h - 2$ $\quad 2x - 6 = h - 2$

$\Rightarrow h = 2x - 4$

@ $x = 2$

$y = -3$

$x^2 - 6x + 5 =$

Commentary

Most candidates answered part (a) well with only a few algebraic slips. In part (b), as shown here, some candidates found the use of differentiation irresistible to help them find the gradient of the curve instead of letting h± 0 in their h – 2.

Mark Scheme

2(a)	When $x = 2$, $y = -3$	B1		PI
	Use of $(2 + h)^2 = 4 + 4h + h^2$	M1		
	Correct method for gradient	M1		
	Gradient $= \dfrac{-3 - 2h + h^2 + 3}{h} = -2 + h$	A2,1	5	A1 if only one small error made
(b)	As h tends to 0,	E2,1		E1 for '$h = 0$'
	... the gradient tends to -2	B1F	3	dependent on at least E1 ft small error in (a)
	Total		**8**	

Question 3

3 The complex number z is defined by

$$z = x + 2\mathrm{i}$$

where x is real.

(a) Find, in terms of x, the real and imaginary parts of:

(i) z^2; *(3 marks)*

(ii) $z^2 + 2z^*$. *(2 marks)*

(b) Show that there is exactly one value of x for which $z^2 + 2z^*$ is real. *(2 marks)*

Student Response

3(a)(i) $z^2 = (x+2i)(x+2i)$

$= x^2 + 4xi - 4$

3(a)(ii) $z^* = x - 2i$

$2z^* = 2x - 4i$

$z^2 + 2z^* = x^2 + 4xi - 4 + 2x - 4i$

$= x^2 + (4i+2)x - (4+4i)$ ✓

3(b) $b^2 - 4ac = 0$

$(4i+2)(4i+2) - 4(-4-4i)$

$= -16 + 16i + 4 + 16 + 16i$

$= 32i + 4$

$= 32i - 4i^2$

$= i(32 - 4i)$ ✗

Commentary

Most candidates found the value of z^2 but, as in this example, did not identify clearly the real and imaginary parts of its value. In part (a)(ii) most found z^* correctly but again did not identify the real and imaginary parts. As this candidate shows to find when $z^2 + 2z^*$ was real instead of letting the imaginary part, which was $4x - 4$, equal zero, candidates looked for a far more complicated solution.

Mark Scheme

3(a)(i)	$z^2 = (x^2 - 4) + i(4x)$	M1A1		M1 for use of $i^2 = -1$
	R and I parts clearly indicated	A1F	3	Condone inclusion of i in I part
				ft one numerical error
(ii)	$z^2 + 2z^* = \left(x^2 + 2x - 4\right) + i\left(4x - 4\right)$	M1A1F	2	M1 for correct use of conjugate
				ft numerical error in (i)
(b)	$z^2 + 2z^*$ real if imaginary part zero	M1		
	... ie if $x = 1$	A1F	2	ft provided imaginary part linear
	Total		7	

Question 4

4 The variables x and y are known to be related by an equation of the form

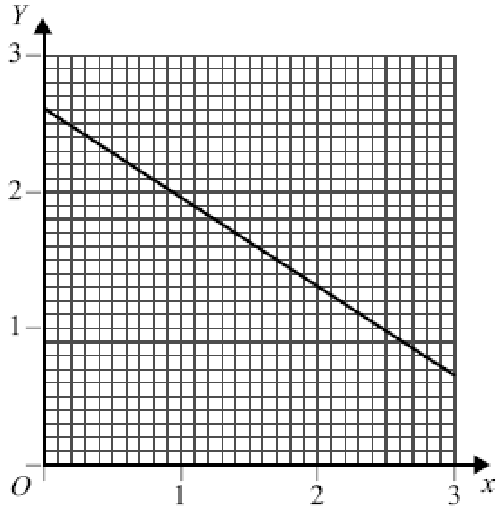
$$y = ab^x$$

where a and b are constants.

(a) Given that $Y = \log_{10} y$, show that x and Y must satisfy an equation of the form

$$Y = mx + c$$ *(3 marks)*

(b) The diagram shows the linear graph which has equation $Y = mx + c$.



Use this graph to calculate:

(i) an approximate value of y when $x = 2.3$, giving your answer to one decimal place;

(ii) an approximate value of x when $y = 80$, giving your answer to one decimal place.

(You are not required to find the values of m and c.) *(4 marks)*

Student Response

4(a) $y = ab^x$ $Y = \log_{10} y$

$Y = x\log b + \log a$

$Y = x\log_{10} b + \log_{10} a$

(b)(i) when $x = 2.3$

$Y = 1.1$

$Y = \log_{10} y$

$1.1 = \log_{10} y$

$y = e^{1.1}$ X

$y = 3.0$ (1 d.p.)

(ii) when $y = 80$

$Y = \log_{10} 80$

$Y = 1.903$

$\therefore x = 1.1$

Commentary

In this example the candidate has just written down the equation $Y = x \log b + \log a$ instead of showing the use of the two log laws and the intermediary result $Y = \log a + \log b^x$. In part (b)(i) this candidate found correctly that 1.1 = log y but then used a power of e rather than 10 to attempt to find y. As shown, part (b)(ii) was usually answered correctly.

Mark Scheme

4(a)	$\lg(ab^x) = \lg a + \lg(b^x)$ $\ldots = \lg a + x \lg b$ Correct relationship established [SC After M0M0, B2 for correct form]	M1 M1 A1	 3	Use of one log law Use of another log law
(b)(i)	When $x = 2.3$, $Y \approx 1.1$, so $y \approx 12.6$	M1A1		Allow 12.7; allow NMS
(ii)	When $y = 80$, $Y \approx 1.90$, so $x \approx 1.1$	M1A1	4	M1 for $Y \approx 1.9$, allow NMS
	Total		7	

Question 5

5 (a) Find the general solution of the equation

$$\cos(3x - \pi) = \frac{1}{2}$$

giving your answer in terms of π. *(6 marks)*

(b) From your general solution, find all the solutions of the equation which lie between 10π and 11π. *(3 marks)*

Student Response

5) $\cos(3x - \pi) = \frac{1}{2}$

$\Rightarrow 3x - \pi = \cos^{-1} \frac{1}{2}$

$3x - \pi = \frac{\pi}{3}$ ✓

$3x = \frac{4\pi}{3}$

$3x = 2N\pi \pm \frac{4\pi}{3}$

$x = \frac{2}{3}N\pi \pm \frac{4\pi}{9}$

solutions $\Rightarrow$ $x = \frac{2}{3}N\pi + \frac{4\pi}{9}$

and

$x = \frac{2}{3}N\pi + \frac{4\pi}{9}$ ✗

where N is an integer

b) $x = \frac{2}{3}(10)\pi + \frac{4\pi}{9} = 22.3^c$

$x = \frac{2}{3}(11)\pi + \frac{4\pi}{9} = 24.4^c$ ✗

Commentary

As shown here, many candidates found just one solution for 3*x* and then attempted to create the general solution. In part (b) candidates often could not use their answer to part (a) to find the roots between 10 π and 11 π; most used n to be 10 and 11 which produced values for x not within the required range.

Mark Scheme

5(a)	$\cos\frac{\pi}{3}=\frac{1}{2}$	B1		Decimals/degrees penalised at 6th mark only
	Appropriate use of $\pm$	B1		OE
	Introduction of $2n\pi$	M1		(or $n\pi$) at any stage
	Going from $3x-\pi$ to x	m1		including dividing all terms by 3
	$x=\frac{\pi}{3}\pm\frac{\pi}{9}+\frac{2}{3}n\pi$	A2,1F	6	OE; A1 with decimals and/or degrees; ft wrong first solution
(b)	At least one value in given range	M1		compatible with c's GS
	Correct values $\frac{92}{9}\pi,\ \frac{94}{9}\pi,\ \frac{98}{9}\pi$	A2,1	3	A1 if one omitted or wrong values included; A0 if only one correct value given
	Total		**9**	

Question 6

6 An ellipse E has equation

$$\frac{x^2}{3}+\frac{y^2}{4}=1$$

(a) Sketch the ellipse E, showing the coordinates of the points of intersection of the ellipse with the coordinate axes. *(3 marks)*

(b) The ellipse E is stretched with scale factor 2 parallel to the y-axis.

Find and simplify the equation of the curve after the stretch. *(3 marks)*

(c) The **original** ellipse, E, is translated by the vector $\begin{bmatrix} a \\ b \end{bmatrix}$. The equation of the translated ellipse is

$$4x^2+3y^2-8x+6y=5$$

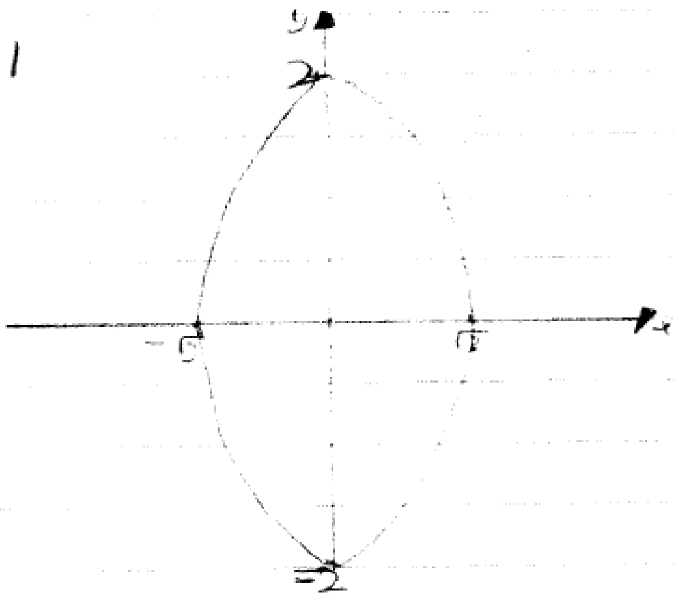
Find the values of a and b. *(5 marks)*

Student Response

6 a $\frac{x^2}{3}+\frac{y^2}{4}=1$

$x=\pm\sqrt{3}$

$y=\pm 2$



b $\frac{x^2}{3}+\frac{\left(\frac{y}{2}\right)^2}{4}=0$

$\frac{x^2}{3}+\frac{y^2}{16}=0$

c $\frac{(x-a)^2}{3}+\frac{(y-b)^2}{4}=1$

$\frac{(x^2-2ax+a^2)}{3}+\frac{(y^2-2by+b^2)}{4}=1$ ✓

$(x^2-2ax+a^2)+3\frac{(y^2-2by+b^2)}{4}=3$

$4(x^2-2ax+a^2)+3(y^2-2by+b^2)=12.$ /

$4x^2+3y^2-8ax-6by+4a^2+3b^2=12.$

compare

$4x^2+3y^2-8x+6y=5$

$4x^2+3y^2=4x^2+3y^2$

$-8ax-6by=-8x+6y \Rightarrow -8a-6b=-8+6 \Rightarrow -8a-6b=-2 \Rightarrow a=\frac{-2+6b}{-8}$

$4a^2+3b^2-12=5 \quad\Rightarrow 4a^2+3b^2=17 \Rightarrow b=\sqrt{\frac{17-4a^2}{3}}$

Substitute $a=\frac{2+6b}{8}$ into $4a^2+3b^2=17$.

$4\left(\frac{2+6b}{8}\right)^2+3b^2=17. \qquad -8a-6\sqrt{\frac{17-4a^2}{3}}=-2.$ X

$\cancel{\frac{8+24b}{8^2}\ 3b^2=} \qquad -8a-6\sqrt{\frac{17}{3}}-6\frac{(2a)}{\sqrt{3}}=-2.$

$4\left(\frac{4+24b+36b^2}{64}\right)+3b^2=17 \quad -8a\sqrt{3}-6\sqrt{17}-12a=-2$

$-8a\sqrt{3}-12a=-2+6\sqrt{17}$

$\frac{4+24b+36b^2}{16}+3b^2=17 \qquad (-8\sqrt{3}-12)a=-2+6\sqrt{17}$

$4+24b+36b^2+48b^2=272.$

$84b^2+24b-268=0$

$21b^2+6b-67=0$

$b=\frac{-6\pm\sqrt{36+5628}}{42}$

$=\frac{-6\pm\sqrt{5664}}{42}=-\frac{1}{7}\pm\frac{\sqrt{5664}}{42}$

Commentary

Parts (a) and (b) were generally answered correctly. In part (c), as shown here, candidates started well and this candidate equated $-8ax - 6by = -8x + 6y$. Instead of writing down a = 1, b = –1 this candidate then became carried away with complicated algebra. This rarely produced the correct solution.

Mark Scheme

6(a)	Ellipse with centre of origin $(\pm\sqrt{3}, 0)$ and (0 ± 2) shown on diagram	B1 B2,1	3	Allow unequal scales on axes Condone AWRT 1.7 for $\sqrt{3}$; B1 for incomplete attempt
(b)	y replaced by $\frac{1}{2}y$ Equation is now $\frac{x^2}{3} + \frac{y^2}{16} = 1$	M1A1 A1	3	M1A0 for $2y$ instead of $\frac{1}{2}y$
(c)	Attempt at completing the square $4(x-1)^2 + 3(y+1)^2 \ldots$ [Alt: replace x by $x - a$ and y by $y - b$ $4x^2 - 8ax + 3y^2 - 6by \ldots$] $a = 1$ and $b = -1$	M1 A1A1 (M1) (m1A1) A1A1	5	M1 if one replacement correct Condone errors in constant terms
	Total		**11**	

Question 7

7 (a) Using surd forms where appropriate, find the matrix which represents:

(i) a rotation about the origin through 30° anticlockwise; *(2 marks)*

(ii) a reflection in the line $y = \dfrac{1}{\sqrt{3}}x$. *(2 marks)*

(b) The matrix $\mathbf{A}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$$

represents a combination of an enlargement and a reflection. Find the scale factor of the enlargement and the equation of the mirror line of the reflection. *(2 marks)*

(c) The transformation represented by $\mathbf{A}$ is followed by the transformation represented by $\mathbf{B}$, where

$$\mathbf{B} = \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$$

Find the matrix of the combined transformation and give a full geometrical description of this combined transformation. *(5 marks)*

Student Response

7. a. i) 30° anticlockwise rotation. $= \begin{bmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{bmatrix}$

$= \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ ✓

ii) reflection in line $y = \frac{1}{\sqrt{3}} x$ $\quad y = \tan\theta\, x$ $\quad \tan\theta = \frac{1}{\sqrt{3}}$

$\theta = 30° \Rightarrow 2\theta = 60°$

$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \rightarrow \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$ ✓

b. $A = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$ Scale factor of enlargement = 2 ✓

Mirror line = reflection in the line $y = \tan 30° x$

$= y = \frac{1}{\sqrt{3}} x$ ✓

c. $\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix} \times \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix} = \begin{bmatrix} (\sqrt{3}) + (\sqrt{3}) & (-1) + (\sqrt{3})^2 \\ (\sqrt{3})^2 + 1 & -\sqrt{3} - \sqrt{3} \end{bmatrix}$

$= \begin{bmatrix} 2\sqrt{3} & 2 \\ 2 & -2\sqrt{3} \end{bmatrix}$

Enlargement of scale factor 4 ✓ followed by a reflection in the line $y = \tan 15° x$

or $y = (2 - \sqrt{3})x$

Commentary

Many candidates used the formulae in the formula booklet to answer correctly part (a). These candidates often correctly identified the transformation in part (b). As shown in this example instead of finding BA in part (c) candidates often found AB.

Mark Scheme

7(a)(i)	Matrix is $\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$	M1A1	2	M1 for $\begin{bmatrix} \cos 30° & \sin 30° \\ -\sin 30° & \cos 30° \end{bmatrix}$ (PI)
(ii)	Matrix is $\begin{bmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$	M1A1	2	M1 for $\begin{bmatrix} \cos 60° & \sin 60° \\ \sin 60° & -\cos 60° \end{bmatrix}$ (PI)
(b)	SF 2, line $y = \frac{1}{\sqrt{3}}x$	B1B1	2	OE
(c)	Attempt at **BA** or **AB**	M1		
	BA = $\begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}$	m1A1		m1 if zeros in correct positions
	Enlargement SF 4	B1F		ft use of **AB** (answer still 4) or after **BA** = $\begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix}$
	... and reflection in line $y = x$	B1F	5	ft only from **BA** = $\begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix}$
	Total		**11**	

Question 8

8 A curve has equation

$$y = \frac{x^2}{(x-1)(x-5)}$$

(a) Write down the equations of the three asymptotes to the curve. *(3 marks)*

(b) Show that the curve has no point of intersection with the line $y = -1$. *(3 marks)*

(c) (i) Show that, if the curve intersects the line $y = k$, then the x-coordinates of the points of intersection must satisfy the equation

$$(k-1)x^2 - 6kx + 5k = 0$$ *(2 marks)*

(ii) Show that, if this equation has equal roots, then

$$k(4k+5) = 0$$ *(2 marks)*

(d) Hence find the coordinates of the two stationary points on the curve. *(5 marks)*

Student Response

8) $x=1$ ✓ $x=5$ ✓ and $y=0$ ✗

b) $-1 = \frac{x^2}{(x-1)(x-5)} \Rightarrow -1 = \frac{x^2}{x^2-6x+5}$

$\Rightarrow -x^2+6x-5 = x^2$

$\Rightarrow -2x^2+6x-5=0$

$\Rightarrow 2x^2-6x+5=0$

$b^2-4ac \geq 0$

$36-40 = -4$ ✓

negative = no roots

c) i) $K = \frac{x^2}{(x-1)(x-5)} \Rightarrow kx^2-6kx+5k=x^2$

$kx^2-6kx+5k-x^2=0$

$(k-1)x^2-6kx+5k=0$ ✓

ii) $b^2-4ac=0$

$36k^2 - (4(k-1)(5)$ ✗

$36k^2 - (20k-20)$

$36k^2-20k-20 = 0$ ✗

$= k(4k+5)=0$

d) $-\frac{5}{4}$ and 0 are x coordinates ✗

Commentary

In part (a), the vertical asymptotes were usually found correctly but often the horizontal asymptote of y = 1 was not. Most candidates knew the appropriate methods to be used in parts (b), (c)(i) and (ii) but often their algebraic skills let them down somewhere, in this case a 'k' term was deleted in part (c)(ii). The answers are given to enable candidates to check that their answers are correct, correct the work if it is not and to enable them to make progress through the next part of the question. Naturally the examiner is watching for inventive algebra as shown in the last line in part (c)(ii) of this script. In part (d) many candidates could make no progress after solving the quadratic found in the previous part.

Mark Scheme

8(a)	Asymptotes $x = 1$, $x = 5$, $y = 1$	B1 × 3	3	
(b)	$y = -1 \Rightarrow (x-1)(x-5) = -x^2$	M1		OE
	$\ldots \Rightarrow 2x^2 - 6x + 5 = 0$	m1		OE
	Disc't = 36 − 40 < 0, so no pt of int'n	A1	3	convincingly shown (AG)
(c)(i)	$y = k \Rightarrow x^2 = k(x^2 - 6x + 5)$	M1		OE
	$\ldots \Rightarrow (k-1)x^2 - 6kx + 5k = 0$	A1	2	convincingly shown (AG)
(ii)	Discriminant $= 36k^2 - 20k(k-1)$	M1		OE
	$\ldots = 0$ when $k(4k+5) = 0$	A1	2	convincingly shown (AG)
(d)	$k = 0$ gives $x = 0, y = 0$	B1		
	$k = -\frac{5}{4}$ gives $-\frac{9}{4}x^2 + \frac{30}{4}x - \frac{25}{4} = 0$	M1A1		OE
	$(3x-5)^2 = 0$, so $x = \frac{5}{3}$	A1		
	$y = -\frac{5}{4}$	B1	5	
	Total		**15**	