# Teacher Support Materials

# Maths GCE

# Paper Reference MFP2

***Question 1***

**1** (a) Express

$$5\sinh x + \cosh x$$

in the form $Ae^{x} + Be^{-x}$, where $A$ and $B$ are integers. *(2 marks)*

(b) Solve the equation

$$5\sinh x + \cosh x + 5 = 0$$

giving your answer in the form $\ln a$, where $a$ is a rational number. *(4 marks)*

***Student Response***

1a) $5 \times \frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2}$ ~~x2~~

$\frac{1}{2}(5e^{x} - 5e^{-x} + e^{x} + e^{-x})$

$= \frac{1}{2}(6e^{x} - 4e^{-x})$

$= 3e^{x} - 2e^{-x}$

$A = 3 \quad B = -2$

Leave blank

2

b) $5\sinh x + \cosh x = -5$

$3e^{x} - 2e^{-x} = -5$

$\ln 3e^{x} - \cancel{\ln 2e}^{-x} = \ln -5$

$x\,\cancel{\ln 3e} - x\ln 2e = \ln -5$

$3x + 2x = \ln -3$

$5x = \ln -5$

$x = \frac{\ln -5}{5}$

$x = \ln -5^{\frac{1}{5}}$

Leave blank

0

2

***Commentary***

The first part of this question was well done by this candidate as indeed by virtually all candidates. However the error in the second part of the question on this script was fairly common. The error basically was to assume that if $p + q + r = 0$ then $\ln p + \ln q + \ln r = 0$ also. This error was made either at the stage of the solution indicated by this script, or sometimes at the stage when the candidate had written $3e^{2x} + 5e^{x} - 2 = 0$.

### *Mark scheme*

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| **1(a)** | $5\left(\frac{e^x - e^{-x}}{2}\right) + \left(\frac{e^x + e^{-x}}{2}\right)$ | M1 | | M0 if no 2s in denominator |
| | $= 3e^x - 2e^{-x}$ | A1 | 2 | |
| **(b)** | $3e^x - 2e^{-x} + 5 = 0$ | | | |
| | $3e^{2x} + 5e^x - 2 = 0$ | M1 | | ft if 2s missing in (a) |
| | $(3e^x - 1)(e^x + 2) = 0$ | A1F | | |
| | $e^x \neq -2$ | E1 | | any **indication** of rejection |
| | $e^x = \frac{1}{3} \qquad x = \ln\frac{1}{3}$ | A1F | 4 | provided quadratic factorises into real factors |
| | **Total** | | **6** | |

### *Question 2*

**2** (a) Given that

$$\frac{1}{r(r+1)(r+2)} = \frac{A}{r(r+1)} + \frac{B}{(r+1)(r+2)}$$

show that $A = \frac{1}{2}$ and find the value of $B$. *(3 marks)*

(b) Use the method of differences to find

$$\sum_{r=10}^{98} \frac{1}{r(r+1)(r+2)}$$

giving your answer as a rational number. *(4 marks)*

### *Student response*

2a) So $\frac{1}{r(r+1)(r+2)} = \frac{A}{r(r+1)} + \frac{B}{(r+1)(r+2)}$

then $1 = A(r+1)(r+2) + B(r(r+1))$. M0

when $r = 0$ $\quad 1 = A(1)(2) + B(0/0)$.

$1 = 2A$

$\underline{A = \frac{1}{2}}$ QED A0

when $r = 1$

$1 = A(2)(3) + B(1)(2)$

$\Rightarrow 1 = 6A + 2B$

$1 = 6 \times \frac{1}{2} + 2B \Rightarrow 2B + 3 = 1 \quad 2B = -2$

Leave blank

$= 2B = -2$

$B = -1$ QED.

2b) $\sum_{r=10}^{98} \frac{1}{r(r+1)(r+2)} = \sum_{r=10}^{98} \frac{1}{2}\left(\frac{1}{r(r+1)}\right) - \frac{1}{(r+1)(r+2)}$

So when $r = 10$ $r = 11$

$\frac{1}{2}\left(\frac{1}{10(11)}\right) - \frac{1}{(11)(12)} + \frac{1}{2}\left(\frac{1}{11(12)}\right) - \frac{1}{12(13)}$

$r = 12$ $\frac{1}{2}\left(\frac{1}{12(13)}\right) - \frac{1}{13(14)}$

No cancellation now

M0

$r = 98$ $r = 97$ $r = 96$

$\frac{1}{2}\left(\frac{1}{98(99)}\right) - \frac{1}{2}\left(\frac{1}{99(100)}\right) + \frac{1}{2}\left(\frac{1}{97(98)}\right) - \frac{1}{98(99)} + \frac{1}{2}\left(\frac{1}{96(97)}\right)$

$- \frac{1}{97(98)}$

So $= \frac{1}{2}\left(\frac{1}{10(11)}\right) + - \frac{1}{99(100)} - \sum_{r=10}^{98} \frac{1}{2}\left(\frac{1}{(r+1)(r+2)}\right)$

0

$= \frac{1}{220} - \frac{1}{9900} - \sum_{r=10}^{98} \frac{1}{2}\left(\frac{1}{(r+1)(r+2)}\right)$

$= \frac{1}{225} - \sum_{r=10}^{98} \frac{1}{2}\left(\frac{1}{(r+1)(r+2)}\right)$

***Commentary***

The error on this script was a very common error. When clearing fractions to find the values of $A$ and $B$, the candidate wrote $1 = A(r+1)(r+2) + Br(r+1)$. Replacing $r$ by zero in this incorrect equation will still give the correct value for $A$, but replacing $r$ by 1 (or sometimes some other value or the equating of coefficients) does not give $B = -\frac{1}{2}$ $(-1 \text{ in this case})$.

In part (b) then, the terms do not cancel out but this does not seem to worry the candidate – he still cancels out and very nearly arrives at the correct answer. In some instances candidates realised that $B$ had to be $-\frac{1}{2}$ even though they were unable to arrive at it in (a), but merely assumed it in part (b).

***Mark Scheme***

| | | | | |
|---|---|---|---|---|
| **2(a)** | $1 = A(r+2) + Br$ | M1 | | |
| | $2A = 1, \quad A = \frac{1}{2}$ | A1 | | |
| | $A + B = 0, \quad B = -\frac{1}{2}$ | A1 | 3 | |
| **(b)** | $r = 10 \quad \frac{1}{2}\left(\frac{1}{10.11} - \frac{1}{11.12}\right)$ | | | if (a) is incorrect but $A = \frac{1}{2}$ and $B = -\frac{1}{2}$ used, allow full marks for (b) |
| | $r = 11 \quad \frac{1}{2}\left(\frac{1}{11.12} - \frac{1}{12.13}\right)$ | | | |
| | $\cdots\cdots\cdot$ | | | |
| | $r = 98 \quad \frac{1}{2}\left(\frac{1}{98.99} - \frac{1}{99.100}\right)$ | M1A1 | | 3 relevant rows seen |
| | $S = \frac{1}{2}\left(\frac{1}{10.11} - \frac{1}{99.100}\right)$ | m1 | | if split into $\frac{1}{2r} - \frac{1}{r+1} + \frac{1}{2(r+2)}$, follow mark scheme, in which case $\frac{1}{2.10} - \frac{1}{2.11} + \frac{1}{2.100} - \frac{1}{2.99}$ scores m1 |
| | $= \frac{89}{19800}$ | A1 | 4 | |
| | **Total** | | 7 | |

***Question 3***

**3** The cubic equation

$$z^3 + qz + (18 - 12\mathrm{i}) = 0$$

where $q$ is a complex number, has roots $\alpha$, $\beta$ and $\gamma$.

(a) Write down the value of:

(i) $\alpha\beta\gamma$; *(1 mark)*

(ii) $\alpha + \beta + \gamma$. *(1 mark)*

(b) Given that $\beta + \gamma = 2$, find the value of:

(i) $\alpha$; *(1 mark)*

(ii) $\beta\gamma$; *(2 marks)*

(iii) $q$. *(3 marks)*

(c) Given that $\beta$ is of the form $k\mathrm{i}$, where $k$ is real, find $\beta$ and $\gamma$. *(4 marks)*

***Student Response***

Leave blank

3 $z^3 + qz + (18 - 12i) = 0$

ai $\alpha\beta\gamma = -(18 - 12i)$

ii $\alpha + \beta + \gamma = 0$

bi $\beta + \gamma = 2$

$\alpha + (\beta + \gamma) = 0 \ \therefore \ \alpha + 2 = 0, \ \alpha = -2$

ii $\alpha\beta\gamma = -(18 - 12i)$

$\alpha = -2, \quad -2\beta\gamma = -(18 - 12i)$ 3

$2\beta\gamma = 18 - 12i$

$\beta\gamma = 9 - 6i$ 2

iii $q = \alpha\beta + \beta\gamma + \gamma\alpha$

$= \beta\gamma + \alpha(\beta + \gamma)$

$= 9 - 6i + (-2)(2)$ 3

$= 5 - 6i$

c $z^3 + (5 - 6i)z + (18 - 12i) = 0$

$(z + 2)(z^2 - 2z + (9 - 6i)) = 0$

~~$z^2 - 2z + 9 - 6i = 0$~~

$(z - 1)^2 - 1 + 9 - 6i = 0$

$(z - 1)^2 = -7 + 6i$

$z = \dfrac{b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{(-2) \pm \sqrt{4 - 4(9 - 6i)}}{2}$

No method

$= \dfrac{-2 \pm \sqrt{36 + 24i}}{2}$ M0

$\beta = \dfrac{-2 + \sqrt{36 + 24i}}{2} \qquad \gamma = \dfrac{-2 + \sqrt{36 + 24i}}{2}$

8

***Commentary***

This candidate performed well in parts (a) and (b) scoring full marks. This was good, as sign errors occurred fairly frequently in these parts of the question. However, this candidate (along with many others) failed to take the hint given in part (c) and ended up with a quadratic equation he was unable to solve. This happened either when $(z+2)$ was taken out as a factor (as happened in this case) or when using $\beta\gamma = 9 - 6\mathrm{i}$ followed by $\beta(2-\beta) = 9 - 6\mathrm{i}$ and by $\beta^2 - 2\beta + 9 - 6\mathrm{i} = 0$. The attempt to solve this equation by the quadratic formula almost universally failed as students were unable to take the square root of a complex number.

### *Mark Scheme*

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| **3(a)(i)** | $\alpha\beta\gamma = -18 + 12\mathrm{i}$ | B1 | 1 | accept $-(18-12\mathrm{i})$ |
| **(ii)** | $\alpha + \beta + \gamma = 0$ | B1 | 1 | |
| **(b)(i)** | $\alpha = -2$ | B1F | 1 | |
| **(ii)** | $\beta\gamma = \dfrac{\alpha\beta\gamma}{\alpha} = 9 - 6\mathrm{i}$ | M1<br>A1F | 2 | ft sign errors in (a) or (b)(i) or slips such as miscopy |
| **(iii)** | $q = \sum \alpha\beta = \alpha(\beta+\gamma) + \beta\gamma$ | M1 | | |
| | $= -2\times 2 + 9 - 6\mathrm{i}$ | A1F | | ft incorrect $\beta\gamma$ or $\alpha$ |
| | $= 5 - 6\mathrm{i}$ | A1F | 3 | |
| **(c)** | $\beta = k\mathrm{i}, \quad \gamma = 2 - k\mathrm{i}$ | B1 | | |
| | $k\mathrm{i}(2 - k\mathrm{i}) = 9 - 6\mathrm{i}$ | M1 | | |
| | $2k = -6 \quad (k^2 = 9) \quad k = -3$ | m1 | | imaginary parts |
| | $\beta = -3\mathrm{i}, \quad \gamma = 2 + 3\mathrm{i}$ | A1 | 4 | |
| | **Total** | | **12** | |

### *Question 4*

**4** (a) A circle $C$ in the Argand diagram has equation

$$|z + 5 - \mathrm{i}| = \sqrt{2}$$

Write down its radius and the complex number representing its centre. *(2 marks)*

(b) A half-line $L$ in the Argand diagram has equation

$$\arg(z + 2\mathrm{i}) = \frac{3\pi}{4}$$

Show that $z_1 = -4 + 2\mathrm{i}$ lies on $L$. *(2 marks)*

(c) (i) Show that $z_1 = -4 + 2\mathrm{i}$ also lies on $C$. *(1 mark)*

(ii) Hence show that $L$ touches $C$. *(3 marks)*

(iii) Sketch $L$ and $C$ on one Argand diagram. *(2 marks)*

(d) The complex number $z_2$ lies on $C$ and is such that $\arg(z_2 + 2\mathrm{i})$ has as great a value as possible.

Indicate the position of $z_2$ on your sketch. *(2 marks)*

### *Student Response*

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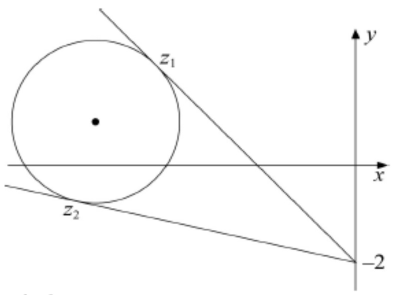
### *Commentary*

This script highlights two important points, one of good practice and one of common error. The good practice is illustrated in part (c)(ii). Generally speaking candidates found this part of the question quite demanding and when it was attempted, it was nearly always attempted using coordinate geometry and leading to some fairly heavy algebra. However, this candidate showed that the gradient of the line from the centre of $C$ to the point representing $z_1$ was $\frac{1}{4}\pi$ and so was perpendicular to the Line $L$ whose gradient was $\frac{3}{4}\pi$. The common error was in part (d) where $z_2$ was placed at the lowest point of the circle vertically below its centre rather than where the second tangent from $(0,-2)$ touched $C$.

### *Mark Scheme*

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| **4(a)** | radius $\sqrt{2}$ centre $-5+\mathrm{i}$ | B1,B1 | 2 | condone $(-5, 1)$ for centre<br>do not accept $(-5, \mathrm{i})$ |
| **(b)** | $\arg(z_1+2\mathrm{i})=\arg(-4+4\mathrm{i})$ | M1 | | |
| | $=\frac{3\pi}{4}$ | A1 | 2 | clearly shown eg $\tan^{-1}\left(-\frac{1}{1}\right)$ |
| **(c)(i)** | $\lvert z_1+5-\mathrm{i}\rvert=\lvert 1+\mathrm{i}\rvert=\sqrt{2}$ | B1 | 1 | |
| **(ii)** | Gradient of line from $(-5, 1)$ to $(-4, 2)$ is 1 $\left(\frac{\pi}{4}\right)$ | M1A1 | | M1 for a complete method |
| | radius $\perp$line $\therefore$ tangent | E1 | 3 | |
| **(iii)** |   | | | |
| | Circle correct | B1F | | ft incorrect centre or radius |
| | Half line correct | B1 | 2 | line must touch $C$ generally above the circle |
| **(d)** | $z_2$ in correct place | B1 | | B0 if $z_2$ is directly below the centre of $C$ |
| | with tangent shown | B1 | 2 | |
| | **Total** | | **12** | |

***Question 5***

**5** (a) Use the definition $\cosh x = \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x})$ to show that $\cosh 2x = 2\cosh^2 x - 1$.

*(2 marks)*

(b) (i) The arc of the curve $y = \cosh x$ between $x = 0$ and $x = \ln a$ is rotated through $2\pi$ radians about the $x$-axis. Show that $S$, the surface area generated, is given by

$$S = 2\pi \int_0^{\ln a} \cosh^2 x \, \mathrm{d}x$$

*(3 marks)*

(ii) Hence show that

$$S = \pi\left(\ln a + \frac{a^4 - 1}{4a^2}\right)$$

*(5 marks)*

***Student Response***

Leave blank

5. a. $\cosh x = \frac{1}{2}(e^{x}+e^{-x})$

$\cosh 2x = \cancel{\cosh x \sinh x + \sinh x \cos}$

$= \cosh^2 x + \sinh^2 x$

$= \left[\frac{1}{2}(e^{x}+e^{-x})\right]^2 + \left[\frac{1}{2}(e^{x}-e^{-x})\right]^2$

$= 2\cosh)$

$= \left(\frac{1}{2}e^{x}+\frac{1}{2}e^{-x}\right)^2 + \left(\frac{1}{2}e^{x}-\frac{1}{2}e^{-x}\right)^2$

$= \left(\frac{1}{4}e^{2x}+\frac{1}{4}e^{-2x}\right) + \frac{1}{4}e^{2x}+\frac{1}{4}e^{-2x}$

$= \frac{1}{4}\left(2e^{2x}+2e^{-2x}\right)$ B0

$= \frac{1}{2}\left(e^{2x}+e^{-2x}\right) = \frac{1}{2}\left[\left(e^{x}+e^{-x}\right)^2 - 2\right]$ B0

$= \frac{1}{2}\left(e^{x}+e^{-x}\right)^2 - 1$

$= 2\cosh^2 x - 1.$

b. i. $y = \cosh x$ between $x=0$ and $x = \ln a$ rotated through $2\pi$ radians about the $x$-axis.

$S = 2\pi \int y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$= 2\pi \int_0^{\ln a} \cosh x \sqrt{1+\sinh^2 x}\, dx.$ B1

$\frac{dy}{dx} = \sinh x \quad \left(\frac{dy}{dx}\right)^2 = \sinh^2 x$ M0

$S = 2\pi \int_0^{\ln a} \cosh x \left(1 + \sinh x\right) dx$ A0

$= 2\pi \int_0^{\ln a} \cosh x + \cosh x \sinh x\, dx$

Leave blank

$$\cancel{S = 2\pi\left[\sinh x + \sinh x\cosh x - \cosh^2 x\right]_0^{\ln a}}$$

$$\cancel{\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx}$$

$$\cancel{\int uv' = uv - \int vu'}$$

$$\cancel{= uv - u\int v}$$

$$\cancel{= \sinh x\cosh x - \cosh^2 x}$$

$$(\cosh x + \cosh x\sinh x)^2 = [\cosh x(1+\sinh x)]^2$$

$$= \cosh^2 x + 2\cosh^2 x\sinh x + \cosh^2 x\sinh^2 x$$

$$= \cosh^2 x(1 + 2\sinh x + \sinh^2 x)$$

$$= \cosh^2 x(\sinh x + 1)(\sinh x + 1)$$

$$= \cosh^2 x(\cosh^2 x - 1 + 1 + 2\sinh x)$$

$$= \quad \cosh^2 x + 2\sinh x$$

$$= \cosh^2 x(1 + 2\sinh x + \sinh^2 x)$$

$$= \sqrt{\cosh^4 x + 2\sinh x\cosh^2 x}$$

$$= \cosh^2 x + \sqrt{2\sinh x\cosh^2 x} = \cosh^2 x.$$

0

Leave blank

ii. $$S = 2\pi\int_0^{\ln a} 1\cdot\cosh^2 x\cdot dx$$

$$= 2\pi\left[x\cosh^2 x - \frac{x^2\cosh^2 x}{2}\right]_0^{\ln a}$$

M0

$$\int uv' = uv - u\int v$$

$$= 2\pi\left[x\cosh^2 x\left(1 - \frac{1}{2}x\right)\right]_0^{\ln a}$$

$$= 2\pi\left(\ln a\tfrac{1}{4}(e^{2\ln a} + e^{-2\ln a})\left(1 - \tfrac{1}{2}\ln a\right)\right) \quad ?$$

$$\cancel{= 2\pi\left(\tfrac{1}{4}\ln a(2a + -2a)\left(1 - \tfrac{1}{2}\ln a\right)\right)}$$

$$\cancel{= 2\pi\left(\tfrac{1}{4}\ln a - \tfrac{1}{8}\ln a\right)}$$

$$=$$

X

X

### *Commentary*

Although some of the errors illustrated here appeared on quite a number of scripts, this particular script seemed to illustrate most of them. Firstly, the direction in part (a) was ignored and the candidate chose instead to quote $\cosh 2x = \cosh^2 x + \sinh^2 x$ thus begging the question. In part (b)(i) although she started off correctly, she made the mistake of trying to run a square root sign through an expression and wrote $\sqrt{1+\sinh^2 x} = 1 + \sinh x$. Finally, in spite of the hint given in part (a) she attempted part (b)(ii) by using integration by parts with no success.

### *Mark Scheme*

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| **5(a)** | $\left(e^x + e^{-x}\right)^2$ expanded correctly | B1 | | $e^{2x} + 2e^0 + e^{-2x}$ is acceptable |
| | Result | B1 | 2 | AG |
| **(b)(i)** | $\frac{dy}{dx} = \sinh x$ | B1 | | |
| | $\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\sinh^2 x}$ | | | |
| | $= \cosh x$ | M1 | | use of $\cosh^2 x - \sinh^2 x = 1$ |
| | $S = 2\pi\int_0^{\ln a} \cosh^2 x \, dx$ | A1 | 3 | AG (clearly derived) |
| **(ii)** | Use of $\cosh^2 x = \frac{1}{2}(1+\cosh 2x)$ | M1 | | allow one slip in formula<br>M0 if $\int \cosh^2 x \, dx$ is given as $\sinh^2 x$ |
| | $S = \pi\left[x + \frac{1}{2}\sinh 2x\right]_0^{\ln a}$ | A1 | | |
| | $= \pi\left[\ln a + \frac{1}{2}\left(\frac{e^{2\ln a} - e^{-2\ln a}}{2}\right)\right]$ | M1 | | |
| | $= \pi\left[\ln a + \frac{1}{4}\left(a^2 - a^{-2}\right)\right]$ | A1F | | |
| | $= \pi\left[\ln a + \frac{1}{4a^2}\left(a^4 - 1\right)\right]$ | A1 | 5 | AG |
| | **Total** | | **10** | |

***Question 6***

**6** By using the substitution $u = x - 2$, or otherwise, find the exact value of

$$\int_{-1}^{5} \frac{\mathrm{d}x}{\sqrt{32 + 4x - x^2}}$$

*(5 marks)*

***Student Response***

6. $\int_{-1}^{5} \sqrt{32+4x-x^2}\ dx \qquad u = x-2 \qquad \frac{du}{dx}\ x = u+2$

$\frac{dx}{du} = 1$ ✓

$= \int_{-1}^{5} \cancel{\frac{1}{\sqrt{32+4(x-2)+(x-2}}}$

$= \int_{-1}^{5} \frac{1}{\sqrt{-(x^2-4x-32)}}$ B1

$= \int \frac{1}{\sqrt{-(x-2)(x-2)+36}}$ B1

$= \int \frac{1}{\sqrt{36-u^2}}\ du \qquad \int_{-1}^{5} u = x-2 \rightarrow \int_{-3}^{3}$

$= \int_{-1}^{5} \frac{1}{\sqrt{6^2-u^2}}\ du$ ✓ $= \left[\sinh \frac{u}{6}\right]_{-3}^{3}$ ✗ M0

$= (\sinh \frac{1}{2}) - (\sinh -\frac{1}{2})$ 0

$= \frac{1}{2}\left(e^{1/2} - e^{-1/2} - e^{-1/2} + e^{1/2}\right)$

$= \frac{1}{2}\left(2e^{1/2} - 2e^{-1/2}\right) = e^{1/2} - e^{-1/2}$ 2

Leave blank

***Commentary***

Although only a relatively short question, responses to it were poor. This particular candidate arrived at $\int \frac{\mathrm{d}u}{\sqrt{6^2 - u^2}}$ and even managed to change the limits correctly. There were, in fact many candidates who were unable to make the substitution correctly. However, what this candidate illustrated, along with many others, a lack of familiarity with the Formulae Booklet provided. $\int \frac{\mathrm{d}x}{\sqrt{a^2 - x^2}}$ (add symbol) in that booklet and candidates should be fully conversant with it so as to make that best possible use of it.

***Mark Scheme***

| | | | | |
|---|---|---|---|---|
| **6** | $u = x - 2$ | | | |
| | $\mathrm{d}u = \mathrm{d}x$ or $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 1$ | B1 | | clearly seen |
| | $32 + 4x - x^2 = 36 - u^2$ | B1 | | if $32 + 4x - x^2$ is written as $36 - (x-2)^2$, give B2 |
| | $\displaystyle\int \frac{\mathrm{d}u}{\sqrt{36 - u^2}} = \sin^{-1}\frac{u}{6}$ | M1 | | allow if d$x$ is used instead of d$u$ |
| | limits −3 and 3<br>or substitute back to give $\sin^{-1}\dfrac{x-2}{6}$ | A1 | | |
| | $\mathrm{I} = \dfrac{\pi}{6} + \dfrac{\pi}{6} = \dfrac{\pi}{3}$ | A1 | 5 | |
| | **Total** | | **5** | |

***Question 7***

**7** (a) Explain why $n(n+1)$ is a multiple of 2 when $n$ is an integer. *(1 mark)*

(b) (i) Given that

$$\mathrm{f}(n) = n(n^2 + 5)$$

show that $\mathrm{f}(k+1) - \mathrm{f}(k)$, where $k$ is a positive integer, is a multiple of 6. *(4 marks)*

(ii) Prove by induction that $\mathrm{f}(n)$ is a multiple of 6 for all integers $n \geqslant 1$. *(4 marks)*

***Student Response***

| | | Leave blank |
|---|---|---|
| 7a/i. | $n(n+1)$ | |
| | ~~at least~~ one of these numbers will ~~be a~~ be not ~~even~~ odd, and and odd number times a number which is not odd gives a non odd number. | E1 |
| bi. | $f(k+1) = (k+1)(k^2+2k+6)$ | |
| | $= k^3+2k^2+6k+k^2+2k+6$ | |
| | $= k^3+3k^2+8k+6$ | |
| | $f(k) = k(k^2+5) = k^3+5k$ | |
| | $f(k+1)-f(k) = k^3-k^3+3k^2+9k-5k+6$ | |
| | $= 3k^2+3k+6$ | M1 |
| | $6M = 3k^2+3k+6 = 6(\frac{1}{2}k^2+\frac{1}{2}k+1)$ | A1 |
| | ~~$M = \frac{1}{2}k^2+\frac{1}{2}k+1$~~ | E0 |
| | $\therefore$ multiple of 6 | |
| ii. | prove for $n=1$ | |
| | $1(1^2+5) = 6 \Rightarrow$ multiple of 6 | B1 |
| | assume true for $n=k$ | |
| | $k(k^2+5) = k^3+5k = 6L$ | |

| | | Leave blank |
|---|---|---|
| | prove true for $n=k+1$ | |
| | $(k+1)((k+1)^2+5)$ | |
| | $=(k+1)(k^2+2k+6)$ | |
| | $=k^3+2k^2+6k+k^2+2k+6$ | |
| | $=k^3+3k^2+8k+6$ | |
| | $=\underbrace{k^3+5k}+3k^2+3k+6$ | M1 |
| | $= 6L + 3k^2+3k+6$ | A1 |
| | $= 6(L+\frac{1}{2}k^2+\frac{1}{2}k+1)$ | E0 |
| | $\therefore$ multiple of 6. | 6 |

***Commentary***

This script illustrated examples of poor reasoning. In part (a) which was managed by most candidates the reason for $n(n+1)$ being even was not convincing. There was no reference to even numbers. A common error was illustrated in part (b)(i) where the candidate wrote $3k^2 + 3k + 6$ followed by $6\left(\frac{1}{2}k^2 + \frac{1}{2}k + 1\right)$ to affirm that $3k^2 + 3k + 6$ was a multiple of 6.

Finally in part (b)(ii) although there was some idea of the method of induction, the solution was completely lacking in formal proof. In order to score full marks on questions on induction a candidate must give a formal proof.

***Mark Scheme***

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| **7(a)** | Clear reason given | E1 | 1 | Minimum O × E = E |
| **(b)(i)** | $(k+1)\left((k+1)^2+5\right)-k\left(k^2+5\right)$ | M1 | | |
| | $=3k^2+3k+6$ | A1 | | |
| | $k^2+k=k(k+1)=M(2)$ | E1 | | **Must** be shown |
| | $\mathrm{f}(k+1)-\mathrm{f}(k)=M(6)$ | E1 | 4 | |
| **(ii)** | Assume true for $n=k$ | | | |
| | $\mathrm{f}(k+1)-\mathrm{f}(k)=M(6)$ | M1 | | Clear method |
| | $\therefore \mathrm{f}(k+1)=M(6)+\mathrm{f}(k)$ | | | |
| | $=M(6)+M(6)$ | A1 | | |
| | $=M(6)$ | | | |
| | True for $n=1$ | B1 | | |
| | $P(n)\rightarrow P(n+1)$ and $\mathrm{P}(1)$ true | E1 | 4 | Provided all other marks earned in (b)(ii) |
| | **Total** | | **9** | |

***Question 8***

**8** (a) (i) Expand

$$\left(z+\frac{1}{z}\right)\left(z-\frac{1}{z}\right) \qquad \textit{(1 mark)}$$

(ii) Hence, or otherwise, expand

$$\left(z+\frac{1}{z}\right)^4\left(z-\frac{1}{z}\right)^2 \qquad \textit{(3 marks)}$$

(b) (i) Use De Moivre's theorem to show that if $z = \cos\theta + \mathrm{i}\sin\theta$ then

$$z^n + \frac{1}{z^n} = 2\cos n\theta \qquad \textit{(3 marks)}$$

(ii) Write down a corresponding result for $z^n - \dfrac{1}{z^n}$. *(1 mark)*

(c) Hence express $\cos^4\theta\sin^2\theta$ in the form

$$A\cos 6\theta + B\cos 4\theta + C\cos 2\theta + D$$

where $A$, $B$, $C$ and $D$ are rational numbers. *(4 marks)*

(d) Find $\displaystyle\int \cos^4\theta\sin^2\theta\ \mathrm{d}\theta$. *(2 marks)*

***Student Response***

Leave blank

8 a) i) $\left(z+\frac{1}{z}\right)\left(z-\frac{1}{z}\right) = z^2 + \frac{z}{z} - \frac{z}{z} - \frac{1}{z^2}$

$= z^2 - \frac{1}{z^2}$ ✓ B 1

ii) $\left(z+\frac{1}{z}\right)^4\left(z-\frac{1}{z}\right)^2 = \left(\left(z+\frac{1}{z}\right)\left(z-\frac{1}{z}\right)\right)^2\left(z+\frac{1}{z}\right)^2$

$= \left(z^2-\frac{1}{z^2}\right)^2\left(z+\frac{1}{z}\right)^2$

$= \left(z^4 - \frac{2z^2}{z^2} + \frac{1}{z^4}\right)\left(z^2 + \frac{2z}{z} + \frac{1}{z^2}\right)$

$= z^6 + 2z^4 + z^2 - 2z^2 - 4 - \frac{2}{z^2} + \frac{1}{z^2} + \frac{2}{z^4} + \frac{1}{z^6}$

$= z^6 + \frac{1}{z^6} + 2\left(z^4 + \frac{1}{z^4}\right) - 2\left(z^2 + \frac{1}{z^2}\right)$

$+ \left(z^2 + \frac{1}{z^2}\right) - 4$

$= z^6 + \frac{1}{z^6} + 2\left(z^4 + \frac{1}{z^4}\right) - \left(z^2 + \frac{1}{z^2}\right) - 4$ ✓ 3

| | | Leave blank |
|---|---|---|
| b) i) | $(\cos\theta + i\sin\theta)^n + \dfrac{1}{(\cos\theta + i\sin\theta)^n}$ | |
| | $= (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$ | |
| | $= \cos n\theta + i\sin n\theta + \cos(-n)\theta + i\sin(-n)\theta$ | 3 |
| | $= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$ | |
| | $= 2\cos n\theta$ | |
| ii) | $z^n - \dfrac{1}{z^n} = 2i\sin n\theta$ B | 1 |
| c) | ~~tot~~ as $\left(z + \frac{1}{z}\right) = 2\cos\theta \qquad \left(z - \frac{1}{z}\right) = 2i\sin\theta$ | |
| | $\left(z + \frac{1}{z}\right)^4 = 16\cos^4\theta \qquad \left(z - \frac{1}{z}\right)^2 = -4\sin^2\theta$ | |
| | $\therefore \left(z + \frac{1}{z}\right)^4\left(z - \frac{1}{z}\right)^2 = -64\cos^4\theta\sin^2\theta$ | |
| | $\therefore \cos^4\theta\sin^2\theta = \dfrac{-1}{64}\left(\left(z^6 + \dfrac{1}{z^6}\right) + 2\left(z^4 + \dfrac{1}{z^4}\right) - \left(z^2 + \dfrac{1}{z^2}\right) - 4\right)$ | |
| | $= \dfrac{-1}{64}\left(2\cos6\theta + 2(2\cos4\theta) - (2\cos2\theta) - 4\right)$ | 4 |
| | $\cos^4\theta\sin^2\theta = \dfrac{-1}{32}\cos6\theta - \dfrac{1}{16}\cos4\theta + \dfrac{1}{32}\cos2\theta + \dfrac{1}{16}$ | |
| d) | $\int\cos^4\theta\sin^2\theta\, d\theta = \int\left(\dfrac{-1}{32}\cos6\theta - \dfrac{1}{16}\cos4\theta + \dfrac{1}{32}\cos2\theta + \dfrac{1}{16}\right)d\theta$ | |
| | $= \dfrac{-1}{192}\sin6\theta - \dfrac{1}{64}\sin4\theta + \dfrac{1}{64}\sin2\theta + \dfrac{1}{16}\theta + c$ | 2 |
| | | 14 |

***Commentary***

Several demonstrations of good practice are shown here. Firstly the candidate takes up the hint of 'Hence' in part (a)(ii) to save herself from considering more algebra than 'otherwise' methods. The grouping of $z^6 + \dfrac{1}{z^6}$, $z^4 + \dfrac{1}{z^4}$ and so on was useful grouping. Part(b)(i) was also completed properly in the sense that the intermediate step $\cos(-n\theta) + \mathrm{i}\sin(-n\theta) = \cos n\theta - \mathrm{i}\sin n\theta$ was clearly shown. In part(c) the 'Hence' was carefully developed with the candidate arriving at $-64\cos^4\theta\sin^2\theta$. (Most candidates merely assumed that this result was $\cos^4\theta\sin^2\theta$). Having used suitable groupings in (a)(i), the candidate found it easy to arrive at the values of $A$,$B$,$C$,and $D$ and finally complete the question correctly.

*Mark Scheme*

| Q | Solution | Marks | Total | Comments |
|---|---|---|---|---|
| **8(a)(i)** | $\left(z+\frac{1}{z}\right)\left(z-\frac{1}{z}\right)=z^2-\frac{1}{z^2}$ | B1 | 1 | |
| **(ii)** | $\left(z^2-\frac{1}{z^2}\right)^2\left(z+\frac{1}{z}\right)^2$ | | | |
| | $=\left(z^4-2+\frac{1}{z^4}\right)\left(z^2+2+\frac{1}{z^2}\right)$ | M1A1 | | Alternatives for M1A1: $\left(z^4+4z^2+6+\frac{4}{z^2}+\frac{1}{z^4}\right)\left(z^2-2+\frac{1}{z^2}\right)$ or $\left(z^3-\frac{1}{z^3}\right)^2-2\left(z^3-\frac{1}{z^3}\right)\left(z-\frac{1}{z}\right)+\left(z-\frac{1}{z}\right)^2$ |
| | $=z^6+\frac{1}{z^6}+2\left(z^4+\frac{1}{z^4}\right)-\left(z^2+\frac{1}{z^2}\right)-4$ | A1 | 3 | CAO (not necessarily in this form) |
| **(b)(i)** | $z^n+\frac{1}{z^n}=\cos n\theta+\mathrm{i}\sin n\theta$ $+\cos(-n\theta)+\mathrm{i}\sin(-n\theta)$ | M1A1 | | |
| | $=2\cos n\theta$ | A1 | 3 | AG<br>SC: if solution is incomplete and $(\cos\theta+\mathrm{i}\sin\theta)^{-n}$ is written as $\cos n\theta-\mathrm{i}\sin n\theta$, award M1A0A1 |
| **(ii)** | $z^n-z^{-n}=2\mathrm{i}\sin n\theta$ | B1 | 1 | |
| **(c)** | RHS $=2\cos 6\theta+4\cos 4\theta-2\cos 2\theta-4$ | M1<br>A1F | | ft incorrect values in (a)(ii) provided they are cosines |
| | LHS $=-64\cos^4\theta\sin^2\theta$ | M1 | | |
| | $\cos^4\theta\sin^2\theta$ $=-\frac{1}{32}\cos 6\theta-\frac{1}{16}\cos 4\theta+\frac{1}{32}\cos 2\theta+\frac{1}{16}$ | A1 | 4 | |
| **(d)** | $-\frac{\sin 6\theta}{192}-\frac{\sin 4\theta}{64}+\frac{\sin 2\theta}{64}+\frac{\theta}{16}\ (+k)$ | M1<br>A1F | 2 | ft incorrect coefficients but not letters $A$, $B$, $C$, $D$ |
| | **Total** | | **14** | |
| | **TOTAL** | | **75** | |