Teacher Support Materials

Maths GCE

Paper Reference MFP3

Question 1

1 (a) Find the value of the constant k for which kx^2e^{5x} is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 25y = 6e^{5x}$$ *(6 marks)*

(b) Hence find the general solution of this differential equation. *(4 marks)*

Student Response

1a) $\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 25y = 6e^{5x}$

P.I : $y = kx^2 e^{5x}$

$\frac{dy}{dx} = 2kxe^{5x} + 5kx^2e^{5x}$

$\frac{d^2y}{dx^2} = 2ke^{5x} + 10kxe^{5x} + 10kxe^{5x} + 25kx^2e^{5x}$

in DE : P.T.O

number

$2ke^{5x} + 20kxe^{5x} + 25kx^2e^{5x} - 20kxe^{5x} - 50kx^2e^{5x} + 25kx^2e^{5x} = 6e^{5x}$

$2ke^{5x} = 6e^{5x}$ $e^{5x} \neq 0$

$2k = 6$

$\underline{k = 3}$

6

Leave blank

Commentary

In general candidates scored very high marks in this question. In part (a) some candidates did not make full use of the given form for the particular integral. In the exemplar the candidate wastes no time, the given form of the particular integral is accurately differentiated twice and the results substituted into the second-order differential equation. Impressively the candidate indicates that the exponential is not equal to zero and so $k = 3$. In part (b) a significant number of candidates did not give their final answer in the form '$y = f(x)$' where $f(x)$ contains two arbitrary constants. In the exemplar the candidate correctly solves the auxiliary equation but then writes down the incorrect complementary function for roots of the auxiliary equation that are real and equal. The candidate clearly shows that the sum of the complementary function and the particular integral leads to the general solution, but the candidate's general solution for the second order differential equation only contains one arbitrary constant instead of the two required. A significant number of candidates failed to include '$y =$ ' in their final answer.

Mark scheme

MFP3

Q	Solution	Marks	Total	Comments
1(a)	$y_{PI} = kx^2e^{5x} \Rightarrow y' = 2kxe^{5x} + 5kx^2e^{5x}$	M1 A1		Product rule to differentiate x^2e^{5x}
	$\Rightarrow y'' = 2ke^{5x} + 10kxe^{5x} + 10kxe^{5x} + 25kx^2e^{5x}$	A1ft		
	$\Rightarrow 2ke^{5x} + 20kxe^{5x} + 25kx^2e^{5x}$ $-10\left(2kxe^{5x} + 5kx^2e^{5x}\right) + 25kx^2e^{5x} = 6e^{5x}$	M1 A1		Substitution into differential equation
	$2k = 6 \Rightarrow k = 3$	A1ft	6	Only ft if xe^{5x} and x^2e^{5x} terms all cancel out
(b)	Aux. eqn. $m^2 - 10m + 25 = 0 \Rightarrow m = 5$	B1		PI
	CF is $(A + Bx)e^{5x}$	M1		
	GS $y = (A + Bx)e^{5x} + 3x^2e^{5x}$	M1 A1ft	4	Their CF + their/our PI ft only on wrong value of k
	Total		**10**	

Question 2

2 The function $y(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y)$$

where $$\mathrm{f}(x, y) = \sqrt{x^2 + y^2 + 3}$$

and $$y(1) = 2$$

(a) Use the Euler formula

$$y_{r+1} = y_r + h\mathrm{f}(x_r, y_r)$$

with $h = 0.1$, to obtain an approximation to $y(1.1)$, giving your answer to four decimal places. *(3 marks)*

(b) Use the improved Euler formula

$$y_{r+1} = y_r + \tfrac{1}{2}(k_1 + k_2)$$

where $k_1 = h\mathrm{f}(x_r, y_r)$ and $k_2 = h\mathrm{f}(x_r + h, y_r + k_1)$ and $h = 0.1$, to obtain an approximation to $y(1.1)$, giving your answer to four decimal places. *(6 marks)*

Student response

2a $x_r = 1,\ y_r = 2,\ h = 0.1$

$2 + 0.1\sqrt{1^2 + 2^2 + 3}$

$= 2.2828$ ✓ — 3

b $k_1 = 0.1\sqrt{8} = 0.2828$ ✓ M1 A1

$k_2 = 0.1\sqrt{(1.1)^2 + 2.2828^2 + 3}$ ✓ M1

$= 0.2071$ ✗ A0

$y(1.1) = 2 + \frac{1}{2}(0.2828 + 0.2071)$ m1 ✓

$= 2.2450$ ✗ A0 — 4

Commentary

Many candidates scored high marks in this question. The most common error occurred in part (b) where k_2 was found incorrectly. In the exemplar the candidate uses the given Euler formula correctly to find the correct approximation to $y(1.1)$. In part (b) the candidate finds k_1 although it would have been safer to show its value to more than the 4d.p. required accuracy for the final answer. The candidate gives a correct numerical expression for k_2 but its evaluation is incorrect. In fact the candidate has worked out the value of $0.1\sqrt{1.1^2 + 0.2828^2 + 3}$. It is worth noting that if the candidate had omitted $0.1\sqrt{1.1^2 + 2.2828^2 + 3}$ from the second line of the solution to (b), the score for part (b) would have only been 2 marks instead of the 4 marks awarded.

Mark Scheme

2(a)	$y_1 = 2 + 0.1\times\sqrt{1^2 + 2^2 + 3}$	M1		
	$y\,(1.1) = 2 + 0.1 \times \sqrt{8}$	A1		
	$y\,(1.1) = 2.28284\ldots = 2.2828$ to 4dp	A1	3	
(b)	$k_1 = 0.1\times\sqrt{8} = 0.2828$	M1 A1ft		PI
	$k_2 = 0.1\times \text{f}\,(1.1,\ 2.2828\ldots)$	M1		
	$= 0.1 \times \sqrt{9.42137\ldots} = 0.3069(425\ldots)$	A1		PI
	$y\,(1.1) = y\,(1) + \frac{1}{2}\left[0.28284\ldots + 0.30694\ldots\right]$	m1		
	2.29489... = 2.2949 to 4dp	A1	6	
	Total		**9**	

Question 3

3 By using an integrating factor, find the solution of the differential equation

$$\frac{dy}{dx} + (\tan x)y = \sec x$$

given that $y = 3$ when $x = 0$. *(8 marks)*

Student Response

3 $\frac{dy}{dx} + (\tan x)y = \sec x$

I.F. $e^{\int \tan x \, dx}$ M1

$= e^{\ln|\sec x|}$ A1

$= \sec x$ A1

$\frac{d}{dx}((\sec x)y) = \sec^2 x$ M1 A1

$y\sec x = \tan x + C$ A1

$y = \sin x + C$

$y=3, x=0$: $3 = 0 + C$ m1

$C = 3$

$y = \sin x + 3$ A0

(9) 7 (7)

Commentary

In general candidates scored very high marks on this question. The most common errors were either forgetting to multiply the right-hand side of the given differential equation by sec *x* or rearranging '*y* sec *x* = tan *x* + *c*' incorrectly as '*y* = tan *x* cos*x* + *c*' before substituting the given boundary conditions. The exemplar illustrates an alternative form, but basically the same type, to the second of these errors. The candidate finds the correct integrating factor, and uses it to find the correct general solution of the first-order differential equation but then makes an error in rearranging '*y* sec *x* = tan *x* + *c*' to '*y* = sin *x* + *c*'. The candidate applies the boundary condition correctly but to an incorrect equation so loses just the last accuracy mark. It is worth noting that if the candidate had applied the boundary condition at an earlier stage, that is, to the equation '*y* sec *x* = tan *x* + *c*' and then rearranged '*y* sec *x* = tan *x* + 3' incorrectly to '*y* = sin *x* + 3' all 8 marks would have been awarded (the examiners would have applied ISW for work after '*y* sec *x* = tan *x* + 3').

Mark Scheme

3	IF is $e^{\int \tan x \, dx}$	M1			
	$= e^{-\ln \cos x} = e^{\ln \sec x}$	A1		Accept either	
	$= \sec x$	A1ft		ft on earlier sign error	
	$\frac{d}{dx}(y \sec x) = \sec^2 x$	M1A1			
	$y \sec x = \int \sec^2 x \, dx$				
	$y \sec x = \tan x + c$	A1		Condone missing c	
	$y = 3$ when $x = 0 \Rightarrow 3 \sec 0 = 0 + c$	m1			
	$c = 3 \Rightarrow y \sec x = \tan x + 3$	A1	8	OE; condone solution finishing at $c = 3$ provided no errors	
	Total		**8**		

Question 4

4 (a) Show that $(\cos\theta + \sin\theta)^2 = 1 + \sin 2\theta$. *(1 mark)*

(b) A curve has cartesian equation

$$(x^2 + y^2)^3 = (x + y)^4$$

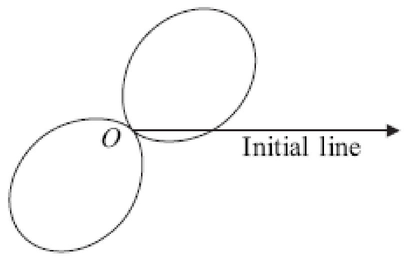
Given that $r \geqslant 0$, show that the polar equation of the curve is

$$r = 1 + \sin 2\theta$$ *(4 marks)*

(c) The curve with polar equation

$$r = 1 + \sin 2\theta, \quad -\pi \leqslant \theta \leqslant \pi$$

is shown in the diagram.



(i) Find the two values of θ for which $r = 0$. *(3 marks)*

(ii) Find the area of one of the loops. *(6 marks)*

Student Response

4. (a) $(\cos\theta+\sin\theta)^2 = \cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta$ ✓

$= 1 + 2\sin\theta\cos\theta$ ✓

$= 1+\sin 2\theta$ ✓

$\because 2\sin\theta\cos\theta = \sin 2\theta$. and $\sin^2\theta+\cos^2\theta = 1$

1

(b) $\because x^2+y^2 = r^2$ ✓

$\therefore (x^2+y^2)^3 = (r^2)^3 = r^6$ ✓

and $\because x = r\cos\theta, \quad y = r\sin\theta$ ✓

$\therefore (x+y)^4 = (r\cos\theta + r\sin\theta)^4 = r^4(\cos\theta+\sin\theta)^4$ ✓

$= r^4((\cos\theta+\sin\theta)^2)^2$ ✓

$= r^4(1+\sin 2\theta)^2$ ✓

$\because (x^2+y^2)^3 = (x+y)^4$

$\therefore r^6 = r^4(1+\sin 2\theta)^2$ ✓

Leave blank

$\Rightarrow r^2 = (1+\sin 2\theta)^2 \quad \because r \geq 0,\ 1+\sin 2\theta \geq 0.$

$\Rightarrow r = 1+\sin 2\theta.$

4

(c)(i) when $r=0$.

$0 = 1+\sin 2\theta.$

$\therefore \sin 2\theta = -1.$

$\therefore 2\theta = -\frac{1}{2}\pi$ or $2\theta = \frac{3}{2}\pi$

$\therefore \theta = -\frac{1}{4}\pi$ or $\theta = \frac{3}{4}\pi.$

3

(ii) $\because A = \frac{1}{2}\int r^2 d\theta = \frac{1}{2}\int_{-\pi}^{\pi}(1+\sin 2\theta)^2 d\theta.$ M1

$= \frac{1}{2}\int_{-\pi}^{\pi}(1+2\sin 2\theta + \sin^2 2\theta)\, d\theta.$ B1

~~$= \frac{1}{2}[\theta - \cos 2\theta$~~

~~$\because \int_b^a \sin^2 2\theta\, d\theta = [\sin 2\theta \cdot (-\frac{1}{2}\cos 2\theta)] - \int -\frac{1}{2}\cos 2\theta \cdot 2\cos 2\theta\, d\theta$~~

$= \frac{1}{2}\int_{-\pi}^{\pi}(1+2\sin 2\theta + \frac{1}{2} - \frac{1}{2}\sin 4\theta)\, d\theta.$ M0

$= \frac{1}{2}\left[\theta - \cos 2\theta + \frac{1}{2}\theta + \frac{1}{8}\cos 4\theta\right]_{-\pi}^{\pi}$

$= \frac{1}{2}(\pi - 1 + \frac{1}{2}\pi + \frac{1}{8} - (-\pi) + 1 + \frac{1}{2}\pi - \frac{1}{8})$

$= \frac{1}{2}\times(3\pi)$

$= \frac{3}{2}\pi.$

2

$\therefore A' = \frac{1}{2}\cdot A = \frac{1}{2}\times\frac{3}{2}\pi = \frac{3}{4}\pi.$

$\therefore$ The area of one of the loops is $\frac{3}{4}\pi$.

(10)

Commentary

In general candidates presented correct answers for parts (a) and (b) although in (b), the method error, $x^2+y^2=r$, was seen more than expected. In general candidates scored at least two marks in part (c)(i). The most common wrong value for θ was $\frac{\pi}{4}$. In the final part of the question most candidates set up an integral with the correct integrand but a significant minority could then not integrate $\sin^2 2\theta$ correctly. Most candidates used their answers to part (c)(i) as the limits but other values were used, not always with any justification. In the exemplar, the candidate shows good examination technique by first quoting the general formula (given on page 8 in the AQA formulae booklet) and then substituting for r rather than going straight to the expanded form. Candidates who failed to show the general formula and started with $A = \frac{1}{2}\int (1+\sin 2\theta)\, d\theta$ scored none of the 6 marks. In the exemplar the candidate uses an incorrect identity '$\sin^2 2\theta = \frac{1}{2}(1 - \sin 4\theta)$' to integrate '$\frac{1}{2}(1 + 2\sin 2\theta + \sin^2 2\theta)$' and although the 'correct' value, $\frac{3\pi}{4}$, for the area of a loop is obtained no further marks can be scored. As explained in the general report, candidates who did not show their method of integrating '$\frac{1}{2}(1 + 2\sin 2\theta + \sin^2 2\theta)$' cannot expect to be awarded any more marks for part (c)(ii) than the marks awarded to the candidate in the exemplar.

Mark Scheme

Q	Solution	Marks	Total	Comments
4(a)	$(\cos\theta + \sin\theta)^2 = \cos^2\theta + \sin^2\theta + 2\cos\theta\sin\theta$ $= 1 + \sin 2\theta$	B1	1	AG (be convinced)
(b)	$(x^2 + y^2)^3 = (x + y)^4$			
	$(r^2)^3 = (r\cos\theta + r\sin\theta)^4$	M2,1,0		[M1 for one of $x^2 + y^2 = r^2$ OE, $x = r\cos\theta$, $y = r\sin\theta$ used]
	$r^6 = r^4 (\cos\theta + \sin\theta)^4$			
	$r^6 = r^4 (1 + \sin 2\theta)^2$	M1		Uses (a) OE at any stage
	$r^2 = (1 + \sin 2\theta)^2$			
	$\Rightarrow r = (1 + \sin 2\theta)\ \{r \geq 0\}$	A1	4	CSO; AG
(c)(i)	$r = 0 \Rightarrow \sin 2\theta = -1$			
	$2\theta = \sin^{-1}(-1);\ = -\frac{\pi}{2}, \frac{3\pi}{2}$	M1		
	$\theta = -\frac{\pi}{4};\ \frac{3\pi}{4}$	A1A1ft	3	A1 for either
(ii)	Area $= \frac{1}{2}\int (1 + \sin 2\theta)^2 \mathrm{d}\theta$	M1		Use of $\frac{1}{2}\int r^2 \mathrm{d}\theta$
	$= \frac{1}{2}\int (1 + 2\sin 2\theta + \sin^2 2\theta)\ \mathrm{d}\theta$	B1		Correct expansion of $(1+\sin 2\theta)^2$
	$= \frac{1}{2}\int \left(1 + 2\sin 2\theta + \frac{1}{2}(1 - \cos 4\theta)\right) \mathrm{d}\theta$	M1		Attempt to write $\sin^2 2\theta$ in terms of $\cos 4\theta$
	$= \left[\frac{3}{4}\theta - \frac{1}{2}\cos 2\theta - \frac{1}{16}\sin 4\theta\right]$	A1ft		Correct integration ft wrong coefficients only
	$= \left[\frac{3}{4}\theta - \frac{1}{2}\cos 2\theta - \frac{1}{16}\sin 4\theta\right]_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}$			
	$= \left(\frac{9\pi}{16}\right) - \left(-\frac{3\pi}{16}\right)$	m1		Using c's values from (c)(i) as limits or the correct limits
	$= \frac{3\pi}{4}$	A1	6	CSO
	Total		**14**	

Question 5

5 (a) A differential equation is given by

$$(x^2 - 1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} = x^2 + 1$$

Show that the substitution

$$u = \frac{dy}{dx} + x$$

transforms this differential equation into

$$\frac{du}{dx} = \frac{2xu}{x^2 - 1}$$ *(4 marks)*

(b) Find the general solution of

$$\frac{du}{dx} = \frac{2xu}{x^2 - 1}$$

giving your answer in the form $u = f(x)$. *(5 marks)*

(c) Hence find the general solution of the differential equation

$$(x^2 - 1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} = x^2 + 1$$

giving your answer in the form $y = g(x)$. *(3 marks)*

Student Response (NEXT PAGE)

number

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5. $u = \frac{dy}{dx} + x$.

$\frac{du}{dx} = \frac{d}{dx}\left(\frac{dy}{dx} + x\right) = \frac{d^2y}{dx^2} + 1$ no.

$\frac{d^2y}{dx^2} - \frac{2x}{(x^2-1)}\frac{dy}{dx} = \frac{x^2+1}{x^2-1}$

$\frac{d^2y}{dx^2} + 1 = \frac{x^2+1}{(x^2-1)} + \frac{2x}{(x^2-1)}\frac{dy}{dx} + 1$

$\frac{d^2y}{dx^2} + 1 = \frac{x^2+1+x^2-1}{x^2-1} + \frac{2x}{x^2-1}\frac{dy}{dx}$

$\frac{d^2y}{dx^2} + 1 = \frac{2x\left(x + \frac{dy}{dx}\right)}{x^2-1}$.

$\frac{du}{dx} = \frac{2xu}{x^2-1}$.

4

b) $\frac{du}{dx} = \frac{2xu}{x^2-1}$

$\frac{du}{u} = \frac{2x}{x^2-1}dx$

$\int \frac{du}{u} = \int \frac{2x}{x^2-1}dx$.

$\ln u = \ln(x^2-1) + c$.

$u = A(x^2-1)$.

5

c). $u = \frac{dy}{dx} + x$

$A(x^2-1) = \frac{dy}{dx} + x$. M1

$\frac{dy}{dx} = A(x^2-1) - x$.

$y = \int Ax^2 - A - x\,dx = \frac{Ax^3}{3} - Ax - \frac{x^2}{2}$. M0 A0

1

10

Commentary

In general candidates gave a correct solution to obtain the printed result in part (a). In part (b), those who used separation of variables normally scored at least 4 of the 5 marks, losing the last mark for writing $u = x^2 - 1 + A$ rather than $u = A(x^2 - 1)$. Those who used an integrating factor approach frequently obtained an incorrect one. In the exemplar the candidate quotes the differential equation to be solved, separates the variables and then integrates both sides correctly to obtain a correct equation involving ln u. The candidate then avoids the common error to impressively reach the correct answer, $u = A(x^2 - 1)$. In part (c) a significant number of candidates formed a correct first-order differential equation involving $\frac{dy}{dx}$ and an arbitrary constant but their general solution to the initial second-order differential equation did not contain two arbitrary constants. This common error is illustrated in the exemplar where the candidate realises that y is obtained by integrating directly but fails to insert the constant of integration thus ending with a general solution to a second-order differential equation where the solution only contains one arbitrary constant.

Mark Scheme

Q	Solution	Marks	Total	Comments
5(a)	$u = \frac{dy}{dx} + x \Rightarrow \frac{du}{dx} = \frac{d^2y}{dx^2} + 1$	M1A1		
	$(x^2-1)\left(\frac{du}{dx} - 1\right) - 2x(u - x) = x^2 + 1$	M1		Substitution into LHS of DE as far as no ys
	DE $\Rightarrow$ $(x^2-1)\frac{du}{dx} - 2xu = 0$			
	$\Rightarrow \frac{du}{dx} = \frac{2xu}{x^2-1}$	A1	4	CSO; AG
(b)	$\int \frac{1}{u}\, du = \int \frac{2x}{x^2-1}\, dx$	M1 A1		Separate variables
	$\ln u = \ln\lvert x^2 - 1\rvert + \ln A$	A1A1		
	$u = A(x^2 - 1)$	A1	5	
(c)	$\frac{dy}{dx} + x = A(x^2 - 1)$	M1		Use (b) $(\neq 0)$ to form DE in y and x
	$\frac{dy}{dx} = A(x^2 - 1) - x$			
	$y = A\left(\frac{x^3}{3} - x\right) - \frac{x^2}{2} + B$	M1 A1ft	3	Solution must have two different constants and correct method used to solve the DE
	Total		**12**	

Question 6

6 (a) The function f is defined by

$$f(x) = \ln(1 + e^x)$$

Use Maclaurin's theorem to show that when $f(x)$ is expanded in ascending powers of x:

(i) the first three terms are

$$\ln 2 + \tfrac{1}{2}x + \tfrac{1}{8}x^2$$ *(6 marks)*

(ii) the coefficient of x^3 is zero. *(3 marks)*

(b) Hence write down the first two non-zero terms in the expansion, in ascending powers of x, of $\ln\left(\dfrac{1 + e^x}{2}\right)$. *(1 mark)*

(c) Use the series expansion

$$\ln(1 + x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \ldots$$

to write down the first three terms in the expansion, in ascending powers of x, of $\ln\left(1 - \dfrac{x}{2}\right)$. *(1 mark)*

(d) Use your answers to parts (b) and (c) to find

$$\lim_{x \to 0}\left[\frac{\ln\left(\dfrac{1 + e^x}{2}\right) + \ln\left(1 - \dfrac{x}{2}\right)}{x - \sin x}\right]$$ *(4 marks)*

Student Response

6ai $f(x) = \ln(1+e^x)$

$f'(x) = \frac{e^x}{1+e^x} = e^x(1+e^x)^{-1}$

$f''(x) = e^x(1+e^x)^{-1} - e^{2x}(1+e^x)^{-2}$

$f(x) = f(0) + x f'(0) + \frac{x^2}{2} f''(0) + \ldots$

$= \ln 2 + x\left(1 \times \frac{1}{2}\right) + \frac{x^2}{2}\left(\frac{1}{2} - \frac{1}{4}\right) + \ldots$

$= \ln 2 + \frac{x}{2} + \frac{x^2}{8} + \ldots$

6

ii $f'''(x) = e^x(1+e^x)^{-1} - e^{2x}(1+e^x)^{-2} - 2e^{2x}(1+e^x)^{-2} + 2e^{3x}(1+e^x)^{-3}$

$f'''(0) = \frac{1}{2} - \frac{1}{4} - \frac{2}{4} + \frac{2}{8}$

$= 0$

So coefficient of $x^3 = 0$

3

b $\ln(1+e^x) = \ln 2 + \frac{x}{2} + \frac{x^2}{8} + \ldots$

$\ln\left(\frac{1+e^x}{2}\right)$

$= \ln\frac{1}{2} + \ln(1+e^x)$

$= \ln\frac{1}{2} + \ln 2 + \frac{x}{2} + \frac{x^2}{8} + \ldots$

$= \frac{x}{2} + \frac{x^2}{8} + \ldots$

1

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c $\ln\left(1-\frac{x}{2}\right) = \frac{-x}{2} - \frac{1}{2}\left(\frac{-x}{2}\right)^2 + \frac{1}{3}\left(\frac{-x}{2}\right)^3 - \ldots$

$= \frac{-x}{2} - \frac{x^2}{8} + \frac{-x^3}{24} - \ldots$ ✓

1

d $\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots$

$x - \sin x = \frac{x^3}{6} - \frac{x^5}{120} - \ldots$ B1

$\lim_{x\to 0}\left(\frac{\ln\left(\frac{1+e^x}{2}\right) + \ln\left(1-\frac{x}{2}\right)}{x - \sin x}\right)$

M1

$= \lim_{x\to 0}\left(\frac{\frac{x}{2} + \frac{x^2}{8} - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{24} + \ldots}{\frac{x^3}{6} - \frac{x^5}{120}}\right)$ M1

$\approx \lim_{x\to 0} \frac{\frac{-x^3}{24}}{\frac{x^3}{6}}$ (because $\frac{x^5}{120} \approx 0$ when $x \to 0$)

$\approx -\frac{1}{4}$ A0 cso

3

14

Commentary

Poor differentiation skills led to many candidates losing a significant number of marks in parts (a). The most common error was $\frac{dy}{dx} = \frac{1}{1+e^x}$. In the exemplar the candidate displays excellent skills in applying the chain rule and product rule with confidence. Candidates generally found part (b) difficult with many obtaining the wrong answer $\frac{1}{4}x + \frac{1}{16}x^2$. In the exemplar the candidate clearly recognises the need to use a law of logarithms and completes the solution within a few lines. Part (c) was generally answered correctly. In general, relatively few candidates scored more than half marks for part (d) although a majority appreciated the need to use at least two terms in the expansion for sin x along with their expansions obtained in parts (b) and (c). In the exemplar the candidate scores the first 3 marks but fails to show the 'reduction' of the numerator and denominator before taking the limit.

The explicit step $\lim_{x \to 0} \dfrac{-\frac{1}{24}+...}{\frac{1}{6}+o(x^2)}$ is missing.

Mark Scheme

Q	Solution	Marks	Total	Comments
6(a)(i)	$f(x)=\ln(1+e^x)$:			
	$f(0)=\ln 2$	B1		
	$f'(x)=\dfrac{e^x}{1+e^x}$ $\quad f'(0)=\dfrac{1}{2}$	M1 A1		Chain rule
	$f''(x)=\dfrac{(1+e^x)e^x-e^xe^x}{(1+e^x)^2}=\dfrac{e^x}{(1+e^x)^2}$	M1 A1		Quotient rule OE
	$f''(0)=\dfrac{1}{4}$			
	so first three terms are:			
	$f(x)=\ln 2+\dfrac{1}{2}x+\dfrac{1}{4}\dfrac{x^2}{2!}=\ln 2+\dfrac{1}{2}x+\dfrac{1}{8}x^2$	A1	6	CSO; AG
(ii)	$f'''(x)=\dfrac{(1+e^x)^2e^x-e^x\left[2(1+e^x)e^x\right]}{(1+e^x)^4}$	M1 A1ft		Chain rule with quotient/product rule ft on $f''(x)=ke^x(1+e^x)^n$ (integer $n<0$)
	$f'''(0)=\dfrac{4-4}{2^4}=0$ {so coefficient of x^3 is zero}	A1	3	CSO; AG; All previous differentiation correct
				SC for those not using Maclaurin's theorem: **maximum** of 4/9
(b)	$\dfrac{1}{2}x+\dfrac{1}{8}x^2$	B1	1	
(c)	$\ln\left(1-\dfrac{x}{2}\right)=$			
	$\left(-\dfrac{x}{2}\right)-\dfrac{1}{2}\left(-\dfrac{x}{2}\right)^2+\dfrac{1}{3}\left(-\dfrac{x}{2}\right)^3-......$	B1	1	
(d)	$\ln\left(\dfrac{1+e^x}{2}\right)+\ln\left(1-\dfrac{x}{2}\right)=-\dfrac{x^3}{24}+...$	M1		Uses previous expansions to obtain first non-zero term of the form kx^3
	$x-\sin x\approx x-\left[x-\dfrac{x^3}{3!}+...\right]\approx\dfrac{x^3}{3!}+...$	B1		
	$\left[\dfrac{\ln\left(\frac{1+e^x}{2}\right)+\ln\left(1-\frac{x}{2}\right)}{x-\sin x}\right]=\dfrac{-\frac{1}{24}x^3+...}{\frac{1}{6}x^3+o(x^5)}$	M1		
	$=\dfrac{-\frac{1}{24}x^3+...}{x^3\left[\frac{1}{6}+o(x^2)\right]}=\dfrac{-\frac{1}{24}+...}{\frac{1}{6}+o(x^2)}$			
	$\lim_{x\to 0}.....=-\dfrac{1}{4}$	A1	4	CSO
	Total		**15**	

Question 7

7 (a) Write down the value of

$$\lim_{x \to \infty} x e^{-x}$$ *(1 mark)*

(b) Use the substitution $u = xe^{-x} + 1$ to find $\int \frac{e^{-x}(1-x)}{xe^{-x}+1}\, dx$. *(2 marks)*

(c) Hence evaluate $\int_1^{\infty} \frac{1-x}{x+e^x}\, dx$, showing the limiting process used. *(4 marks)*

Student Response

Question number | Leave blank

7)a) 0 ✓

b) $u = xe^{-x} + 1$

$du = e^{-x} - xe^{-x}\, dx$

$du = e^{-x}(1-x)\, dx$

$\Rightarrow \int \frac{e^{-x}(1-x)\, dx}{xe^{-x}+1} = \int \frac{du}{u}$

$= \ln u + c = \ln|xe^{-x}+1| + c$

2

c) consider $I = \int_1^a \frac{1-x}{x+e^x}\, dx$

~~Let $u = x + e^x$~~
~~$du = 1 + e^x\, dx$~~

WR

~~$\int_1^a \frac{1-x}{u} \times \frac{1}{1+e^x}\, du$~~

Multiply through by e^{-x}

$I = \int_1^a \frac{e^{-x}(1-x)}{xe^{-x}+e^xe^{-x}}\, dx = \int_1^a \frac{e^{-x}(1-x)}{xe^{-x}+1}\, dx$

~~$= \left[\ln|xe^{-x}+1|\right]_1^a = \ln|ae^{-a}+1| - \ln|2|$~~

~~$\int_1^\infty \frac{1-x}{x+e^x}\, dx = \lim_{a\to\infty}\left[\ln|ae^{-a}+1| - \ln|2|\right]$~~

~~$= -\ln 2$~~

~~as $\lim_{a\to\infty}(ae^{-a}) = 0$ so $\ln|ae^{-a}+1| = 0$.~~

WR

Leave blank

7) c) cont

$$I = \int_1^a \frac{e^{-x}(1-x)}{xe^{-x}+1}\,dx$$

$$= \Big[\ln|xe^{-x}+1|\Big]_1^a = \ln|ae^{-a}+1| - \ln|e^{-1}+1|$$

$$\text{so } \int_1^\infty \frac{1-x}{x+e^x}\,dx = \lim_{a\to\infty} I$$

$$= \lim_{a\to\infty}\left(\ln|ae^{-a}+1| - \ln|e^{-1}+1|\right)$$

$$= -\ln|e^{-1}+1|$$

$$\text{as } \lim_{a\to\infty}(ae^{-a}) = 0$$

~~$= \ln\left(\frac{1}{e}+1\right)$~~ ~~$\ln\left(\frac{e}{1+e}\right)$~~ ~~$= \ln e - \ln|1+e|$~~

WR ~~$= -\ln\left|\frac{1}{1+e}\right|$~~

4 / 7

Commentary

In general part (a) was answered correctly but final answers to part (b) were not always given in terms of *x*. In the exemplar, the candidate states the correct answer to part (a) and sets up the correct details for applying the method of substitution to move from variable *x* to variable *u*. The candidate then integrates and substitutes back to give the correct answer as a function of *x*. In general, candidates did not score full marks for part (c). There were a significant number of candidates did not explicitly show the link between the integrand in (c) and the integrand in (b) and the limiting process was not always shown convincingly. In the exemplar, the candidate clearly shows the link between (b) and (c) by multiplying the numerator and denominator of $\frac{1-x}{x+e^x}$ by e^{-x}. The candidate uses part (b) to find the integral and substitutes the limits *a* and 1 to reach $\ln\left(ae^{-a}+1\right) - \ln(e^{-1}+1)$. The candidate then considers the limit as *a* tend to ∞, using the answer to part (a).

The candidate clearly shows the limiting process by

- considering the integral with limit *a* replacing ∞,
- considering the $\lim_{a\to\infty}$ of $[\ln\left(ae^{-a}+1\right) - \ln(e^{-1}+1)]$,
- stating clearly that $\lim_{a\to\infty}(ae^{-a}) = 0$

Mark Scheme

MFP3 (cont)

Q	Solution	Marks	Total	Comments
7(a)	0	B1	1	
(b)	$u = xe^{-x} + 1 \Rightarrow du = \left(e^{-x} - xe^{-x}\right)dx$	M1		Attempts to find du
	$\int \frac{e^{-x}(1-x)}{xe^{-x}+1}\,dx = \int \frac{1}{u}\,du = \ln u + c$			
	$= \ln\left(xe^{-x}+1\right)\{+c\}$	A1	2	Condone missing c
(c)	$\int \frac{1-x}{x+e^{x}}\,dx = \int \frac{e^{-x}(1-x)}{xe^{-x}+1}dx$	B1		
	$\int_1^{\infty} \frac{1-x}{x+e^{x}}\,dx = \lim_{a\to\infty}\left[\ln(xe^{-x}+1)\right]_1^a$			
	$= \lim_{a\to\infty}\left\{\ln\left(ae^{-a}+1\right)\right\} - \ln\left(e^{-1}+1\right)$	M1		For using part (b) and $F(B)-F(A)$
	$= \ln\left\{\lim_{a\to\infty}\left(ae^{-a}+1\right)\right\} - \ln\left(e^{-1}+1\right)$			
	$= \ln 1 - \ln\left(e^{-1}+1\right) = -\ln\left(e^{-1}+1\right)$	M1 A1	4	For using limiting process
	Total		7	