MS03 Support Material

Differences of two independent Poisson means

For use with AQA A-level Mathematics Specification (6360)

1 Outline Theory

Assume that X_A is the number of occurrences of event A during a unit interval and that it may be modelled by a Poisson distribution with mean λ_A.

Assume that X_B is the number of occurrences of event B during a unit interval and that it may be modelled by a Poisson distribution with mean λ_B.

Assume that X_A and X_B are independent.

Suppose that there are R_A occurrences of event X_A during an interval of n_A units.

Suppose that there are R_B occurrences of event X_B during an interval of n_B units.

Then $\bar{X}_A = \dfrac{R_A}{n_A}$ is distributed approximately as $N\left(\lambda_A, \dfrac{\lambda_A}{n_A}\right)$

and $\bar{X}_B = \dfrac{R_B}{n_B}$ is distributed approximately as $N\left(\lambda_B, \dfrac{\lambda_B}{n_B}\right)$.

Thus, using $\dfrac{\bar{x}_A}{n_A}$ and $\dfrac{\bar{x}_B}{n_B}$ as estimates of $\dfrac{\lambda_A}{n_A}$ and $\dfrac{\lambda_B}{n_B}$ respectively, an **approximate** confidence interval (CI) for $\lambda_A - \lambda_B$ is given by:

$$\left(\bar{x}_A - \bar{x}_B\right) \pm z \times \sqrt{\frac{\bar{x}_A}{n_A} + \frac{\bar{x}_B}{n_B}} .$$

Also an **approximate** test statistic (TS) for testing: H_0: $\lambda_A - \lambda_B = a \ (\neq 0)$

is given by:

$$z = \frac{\left(\bar{x}_A - \bar{x}_B\right) - a}{\sqrt{\dfrac{\bar{x}_A}{n_A} + \dfrac{\bar{x}_B}{n_B}}} .$$

When $a = 0$, the two estimates can be pooled by $\bar{X}_P = \dfrac{R_A + R_B}{n_A + n_B}$ and hence an **approximate** test statistic is given by:

$$z = \frac{\left(\bar{x}_A - \bar{x}_B\right)}{\sqrt{\bar{x}_P \times \left(\dfrac{1}{n_A} + \dfrac{1}{n_B}\right)}} .$$

Notes

- The normal approximations assume that either λ_A and λ_B are at least 15 (λ large) or that n_A and n_B are at least 30 (Central Limit Theorem) or both.

- The above hold for $n_A = n_B = 1$ providing λ_A and λ_B are at least 15 (λ large) and that occurrences are counted during intervals of equal width.

2 Examples (In the examination, questions will usually have a context)

Question 1

The Poisson event A occurs 704 times during an interval of 200 units. The Poisson event B occurs 444 times during an interval of 150 units.

Given that the events A and B are independent with unit means λ_A and λ_B respectively, construct an approximate 95% confidence interval for $\lambda_A - \lambda_B$.

Solution 1

Here $\bar{x}_A = \frac{704}{200} = 3.52$ with $n_A = 200$

and $\bar{x}_B = \frac{444}{150} = 2.96$ with $n_B = 150$

A 95% CI implies that $z = 1.96$ so the CI is given by:

$$(3.52 - 2.96) \pm 1.96 \times \sqrt{\frac{3.52}{200} + \frac{2.96}{150}}$$

ie $\underline{(0.18,\ 0.94)}$

Question 2

During 100 unit intervals, there are 387 occurrences of X_1 which has a Poisson distribution with parameter λ_1 per unit interval.

During 200 unit intervals, there are 626 occurrences of X_2 which has a Poisson distribution with parameter λ_2 per unit interval.

Assuming that X_1 and X_2 are independently distributed, investigate, at the 1% level of significance, the claim that λ_1 exceeds λ_2 by more than 0.25.

Solution 2

H_0: $\lambda_1 - \lambda_2 = 0.25$
H_1: $\lambda_1 - \lambda_2 > 0.25$

SL $\alpha = 0.01$
CV $z = 2.3263$

TS $$z = \frac{(3.87 - 3.13) - 0.25}{\sqrt{\frac{3.87}{100} + \frac{3.13}{200}}} = \frac{0.49}{0.23313} = \underline{2.10}$$

Thus insufficient evidence, at the 1% level of significance, to suggest that λ_1 exceeds λ_2 by more than 0.25.

Question 3

During 120 unit intervals, there are 29 occurrences of Y_A which has a Poisson distribution with parameter λ_A per unit interval.

During 90 unit intervals, there are 34 occurrences of Y_B which has a Poisson distribution with parameter λ_B per unit interval.

Assuming that Y_A and Y_B are independently distributed, test, at the 10% level of significance, the claim that λ_A equals λ_B.

Solution 3

H_0: $\lambda_A - \lambda_B = 0$
H_1: $\lambda_A - \lambda_B \neq 0$

SL $\alpha = 0.10$
CV $z = \pm 1.6449$

Pooled estimate $= \dfrac{29+34}{120+90} = \underline{0.3}$

TS $$z = \frac{(0.24167 - 0.37778)}{\sqrt{0.3\left(\frac{1}{210} + \frac{1}{90}\right)}} = \frac{-0.13611}{0.07638} = \underline{-1.78}$$

Thus sufficient evidence, at the 10% level of significance, to suggest that the claim λ_A equals λ_B is false.

Notes

Without pooling TS $$z = \frac{(0.24167 - 0.37778)}{\sqrt{\frac{0.24167}{210} + \frac{0.37778}{90}}} = \frac{-0.13611}{0.07313} = \underline{-1.86}$$