MS03 Support Material

Proofs of mean and variance of binomial and Poisson distributions

For use with AQA A-level Mathematics Specification (6360)

1 Binomial

1.1 Mean

Mean = $\mathrm{E}(X) = \mu = \sum x_i p_i$ where $\sum p_i = 1$	Definition
$= \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=1}^{n} x \binom{n}{x} p^x (1-p)^{n-x}$	Value is zero at $x = 0$
$= \sum_{x=1}^{n} x \dfrac{n!}{x!\,(n-x)!} p^x (1-p)^{n-x}$	Definition of $\binom{n}{x}$
$= np \sum_{x=1}^{n} \dfrac{(n-1)!}{(x-1)!\,(n-x)!} p^{x-1} (1-p)^{n-x}$	Take out factor of np Cancel x as $x! = x(x-1)!$
$= np \sum_{y=0}^{m} \dfrac{m!}{y!(m-y)!} p^y (1-p)^{m-y}$	Substitution of $y = x - 1$ Upper limit of summation is then $(n - 1)$ Substitution of $m = n - 1$ Noting $(n - x) = (m - y)$
$= np \times \sum p_i = np \times 1 = \underline{np}$	Sum of all binomial terms = sum of all probabilities = 1

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1.2 Variance

Variance = $\mathrm{E}(X^2) - (\mathrm{E}(X))^2 = \sigma^2 = \sum x_i^2 p_i - \mu^2$ where $\sum p_i = 1$	Definition
(a) Firstly consider $\mathrm{E}(X(X-1)) = \mathrm{E}(X^2) - \mathrm{E}(X) = \mathrm{E}(X^2) - \mu$	Expansion
$= [\mathrm{E}(X^2) - \mu^2] + \mu^2 - \mu$	Add & subtract $\mu^2 = (\mathrm{E}(X))^2$
$= \sigma^2 + \mu^2 - \mu$	Substitution
Thus $\sigma^2 = \mathrm{E}(X(X-1)) - \mu^2 + \mu = \underline{\mathrm{E}(X(X-1)) - n^2p^2 + np}$	Rearranging & substitution of $\mu = np$
(b) Secondly reconsider $\mathrm{E}(X(X-1))$	
$= \sum_{x=0}^{n} x(x-1)\binom{n}{x} p^x (1-p)^{n-x}$	Definition
$= \sum_{x=2}^{n} x(x-1)\binom{n}{x} p^x (1-p)^{n-x}$	Value is zero at $x = 0$ and 1
$= \sum_{x=2}^{n} x(x-1)\dfrac{n!}{x!\,(n-x)!}\, p^x (1-p)^{n-x}$	Definition of $\binom{n}{x}$
$= n(n-1)\,p^2 \sum_{x=2}^{n} \dfrac{(n-2)!}{(x-2)!\,(n-x)!}\, p^{x-2}\,(1-p)^{n-x}$	Take out factor of $n(n-1)p^2$ Cancel $x(x-1)$ as $x! = x(x-1)(x-2)!$
$= n(n-1)\,p^2 \sum_{y=0}^{m} \dfrac{m!}{y!(m-y)!}\, p^y\,(1-p)^{m-y}$	Substitution of $y = x - 2$ Upper limit of summation is then $(n - 2)$ Substitution of $m = n - 2$ Noting $(n - x) = (m - y)$
$= n(n-1)p^2 \times \sum p_i = n(n-1)p^2 \times 1 = \underline{n(n-1)p^2}$	Sum of all binomial terms = sum of all probabilities = 1
(c) Finally, using parts (a) and (b)	
$\sigma^2 = \mathrm{E}(X(X-1)) - n^2p^2 + np = n(n-1)p^2 - n^2p^2 + np$	
$= n^2p^2 - np^2 - n^2p^2 + np$	Expanding
$= np - np^2 = \underline{np(1-p)}$	Cancelling & factorising

2 Poisson

2.1 Mean

Mean = $\mathrm{E}(X) = \mu = \sum x_i p_i$ where $\sum p_i = 1$ — Definition

$$= \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=1}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!}$$

Value is zero at $x = 0$

$$= \lambda \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!} = \lambda \sum_{y=0}^{\infty} \frac{e^{-\lambda}\lambda^{y}}{y!}$$

Take out factor of λ
Cancel x as $x! = x(x-1)!$
Substitution of $y = x - 1$

$$= \lambda \times \sum p_i = \lambda \times 1 = \underline{\lambda}$$

Sum of all Poisson terms = sum of all probabilities = 1

2.2 Variance

Variance = $\mathrm{E}(X^2) - (\mathrm{E}(X))^2 = \sigma^2 = \sum x_i^2 p_i - \mu^2$ where $\sum p_i = 1$ — Definition

(a) Firstly consider $\mathrm{E}(X(X-1)) = \mathrm{E}(X^2) - \mathrm{E}(X) = \mathrm{E}(X^2) - \mu$ — Expansion

$$= [\mathrm{E}(X^2) - \mu^2] + \mu^2 - \mu$$

Add & subtract $\mu^2 = (\mathrm{E}(X))^2$

$$= \sigma^2 + \mu^2 - \mu$$

Substitution

Thus $\sigma^2 = \mathrm{E}(X(X-1)) - \mu^2 + \mu = \underline{\mathrm{E}(X(X-1)) - \lambda^2 + \lambda}$

Rearranging & substitution of $\mu = \lambda$

(b) Secondly reconsider $\mathrm{E}(X(X-1))$

$$= \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=2}^{\infty} x(x-1) \frac{e^{-\lambda}\lambda^x}{x!}$$

Definition
Value is zero at $x = 0$ and 1

$$= \lambda^2 \sum_{x=2}^{\infty} \frac{e^{-\lambda}\lambda^{x-2}}{(x-2)!} = \lambda^2 \sum_{y=0}^{\infty} \frac{e^{-\lambda}\lambda^{y}}{y!}$$

Take out factor of λ^2
Cancel $x(x-1)$ as $x! = x(x-1)(x-2)!$
Substitution of $y = x - 2$

$$= \lambda^2 \times \sum p_i = \lambda^2 \times 1 = \underline{\lambda^2}$$

Sum of all Poisson terms = sum of all probabilities = 1

(c) Finally, using parts (a) and (b)

$$\sigma^2 = \mathrm{E}(X(X-1)) - \lambda^2 + \lambda = \lambda^2 - \lambda^2 + \lambda$$

Expanding

$$= \underline{\lambda}$$

Cancelling

3 Some alternative approaches

3.1 Binomial

The following results may be quoted/used in the proofs.

Probability generating function $G(t) = E(t^X) = (q + pt)^n$ where $q = 1 - p$

Moment generating function $M(t) = E(e^{tX}) = (q + pe^t)^n$ where $q = 1 - p$

The following properties may then be used to find the mean and variance.

$$\mu = \left.\frac{dG(t)}{dt}\right|_{t=1} \quad \text{or} \quad \mu = \left.\frac{dM(t)}{dt}\right|_{t=0}$$

$$\sigma^2 = \left.\frac{d^2G(t)}{d^2t}\right|_{t=1} + \mu - \mu^2 \quad \text{or} \quad \sigma^2 = \left.\frac{d^2M(t)}{d^2t}\right|_{t=0} - \mu^2$$

3.2 Poisson

The following results may be quoted/used in the proofs.

Probability generating function $G(t) = E(t^X) = e^{\lambda t - \lambda}$

Moment generating function $M(t) = E(e^{tX}) = e^{\lambda e^t - \lambda}$

The following properties may then be used to find the mean and variance.

$$\mu = \left.\frac{dG(t)}{dt}\right|_{t=1} \quad \text{or} \quad \mu = \left.\frac{dM(t)}{dt}\right|_{t=0}$$

$$\sigma^2 = \left.\frac{d^2G(t)}{d^2t}\right|_{t=1} + \mu - \mu^2 \quad \text{or} \quad \sigma^2 = \left.\frac{d^2M(t)}{d^2t}\right|_{t=0} - \mu^2$$