MS04 Support Material

Proofs of mean and variance of geometric and exponential distributions

For use with AQA A-level Mathematics Specification (6360)

1 Geometric

1.1 Useful results

The following results, for $|x| < 1$, may be quoted/used in the proofs.

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \ldots \quad ①$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots \quad ②$$

$$(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + 15x^4 + \ldots \quad ③$$

1.2 Mean

Mean = $\mathrm{E}(X) = \mu = \sum x_i p_i$ where $\sum p_i = 1$	Definition
$= \sum_{x=1}^{\infty} xp(1-p)^{x-1} = \sum_{x=1}^{\infty} xpq^{x-1}$	Replace $(1 - p)$ by q
$= p + 2pq + 3pq^2 + 4pq^3 + 5pq^4 + \ldots$	Expand summation
$= p(1 + 2q + 3q^2 + 4q^3 + 5q^4 + \ldots)$	Take out factor of p
$= p(1 - q)^{-2}$	Using result ② above
$= p \times p^{-2}$	Substitution for $(1 - q) = p$
$= \underline{p^{-1}}$ or $\underline{\dfrac{1}{p}}$	Simplification

1.3 Variance (method 1)

Variance = $E(X^2) - (E(X))^2 = \sigma^2 = \sum x_i^2 p_i - \mu^2$ where $\sum p_i = 1$	Definition
(a) Firstly consider $E(X(X-1)) = E(X^2) - E(X) = E(X^2) - \mu$	Expansion
$= [E(X^2) - \mu^2] + \mu^2 - \mu$	Add & subtract $\mu^2 = (E(X))^2$
$= \sigma^2 + \mu^2 - \mu$	Substitution
Thus $\sigma^2 = E(X(X-1)) - \mu^2 + \mu = \underline{E(X(X-1)) - p^{-2} + p^{-1}}$	Rearranging & substitution of $\mu = p^{-1}$
(b) Secondly reconsider $E(X(X-1))$	
$= \sum_{x=1}^{\infty} x(x-1)p(1-p)^{x-1}$	Definition
$= \sum_{x=1}^{\infty} x(x-1)pq^{x-1}$	Replace $(1-p)$ by q
$= \sum_{x=2}^{\infty} x(x-1)pq^{x-1}$	Value is zero at $x = 1$
$= 2pq + 6pq^2 + 12pq^3 + 20pq^4 + 30pq^5 + \ldots$	Expand summation
$= 2pq(1 + 3q + 6q^2 + 10q^3 + 15q^4 + \ldots)$	Take out factor of $2pq$
$= 2pq(1-q)^{-3}$	Using result ③ from previous page
$= 2pq \times p^{-3}$	Substitution for $(1-q) = p$
$= \underline{2qp^{-2}}$ or $\underline{\dfrac{2q}{p^2}}$	Simplification
(c) Finally, using parts (a) and (b)	
$\sigma^2 = E(X(X-1)) - p^{-2} + p^{-1} = \dfrac{2q}{p^2} - \dfrac{1}{p^2} + \dfrac{1}{p}$	
$= \dfrac{2(1-p)}{p^2} - \dfrac{1}{p^2} + \dfrac{1}{p}$	Substitution for $q = (1-p)$
$= \dfrac{2-2p-1+p}{p^2}$	Expanding and using a common denominator of p^2
$= \underline{\dfrac{1-p}{p^2}}$	Simplification

1.4 Variance (method 2)

Variance = $E(X^2) - (E(X))^2 = \sigma^2 = \sum x_i^2 p_i - \mu^2$ where $\sum p_i = 1$	Definition
Now $E(X^2) = \sum_{x=1}^{\infty} x^2 p(1-p)^{x-1}$	Definition
$= \sum_{x=1}^{\infty} x^2 pq^{x-1}$	Replace $(1 - p)$ by q
$= p + 4pq + 9pq^2 + 16pq^3 + 25pq^4 + \ldots$	Expand summation
$= (p + 3pq + 6pq^2 + 10pq^3 + 15pq^4 + \ldots)$ $+ (pq + 3pq^2 + 6pq^3 + 10pq^4 + \ldots)$	Split into 2 series
$= p(1 + 3q + 6q^2 + 10q^3 + 15q^4 + \ldots)$	Take out factor of p
$+ pq(1 + 3q + 6q^2 + 10q^3 + \ldots)$	Take out factor of pq
$= p(1-q)^{-3} + pq(1-q)^{-3}$	Using result ③ from previous page
$= \frac{p}{p^3} + \frac{pq}{p^3}$	Substitution for $(1 - q) = p$
$= \frac{1}{p^2} + \frac{q}{p^2} = \frac{1}{p^2} + \frac{1-p}{p^2} = \underline{\frac{2-p}{p^2}}$	Simplification and substitution for $q = (1 - p)$
Thus variance = $E(X^2) - (E(X))^2$	
$= \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2$	Substitution
$= \underline{\frac{1-p}{p^2}}$	Simplification

2 Exponential

2.1 Mean

Mean = $E(X) = \mu = \int_0^\infty x\,\lambda e^{-\lambda x} dx$ where $\int_0^\infty \lambda e^{-\lambda x} dx = 1$	Definition
$= \left[(x)\left(-e^{-\lambda x}\right)\right]_0^\infty - \int_0^\infty (1)\left(-e^{-\lambda x}\right) dx$	Integration by parts
$= [0-0] + \int_0^\infty e^{-\lambda x}\,dx = \frac{1}{\lambda}\int_0^\infty \lambda e^{-\lambda x}\,dx$	Substitution of limits Multiply & divide by λ
$= \frac{1}{\lambda}\times 1 = \underline{\frac{1}{\lambda}}$	Definition of PDF

2.2 Variance

Variance = $E(X^2) - (E(X))^2 = \sigma^2$	Definition
Now $E(X^2) = \int_0^\infty x^2 \lambda e^{-\lambda x}\,dx$	
$= \left[\left(x^2\right)\left(-e^{-\lambda x}\right)\right]_0^\infty - \int_0^\infty (2x)\left(-e^{-\lambda x}\right) dx$	Integration by parts
$= [0-0] + 2\int_0^\infty x e^{-\lambda x}\,dx = \frac{2}{\lambda}\int_0^\infty x\lambda e^{-\lambda x}\,dx$	Substitution of limits Multiply & divide by λ
$= \frac{2}{\lambda}\times E(X) = \frac{2}{\lambda}\times\frac{1}{\lambda} = \frac{2}{\lambda^2}$	Definition and value of $E(X)$
Thus variance = $E(X^2) - (E(X))^2$	
$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2$	Substitution
$= \underline{\frac{1}{\lambda^2}}$	Simplification

3 Some alternative approaches

3.1 Geometric

The following results may be quoted/used in the proofs.

Probability generating function $\mathrm{G}(t) = \mathrm{E}(t^X) = \dfrac{pt}{1-qt}$ where $q = 1 - p$

Moment generating function $\mathrm{M}(t) = \mathrm{E}(\mathrm{e}^{tX}) = \dfrac{p\mathrm{e}^t}{1-q\mathrm{e}^t}$ where $q = 1 - p$

The following properties may then be used to find the mean and variance.

$$\mu = \left.\frac{\mathrm{d}\,\mathrm{G}(t)}{\mathrm{d}t}\right|_{t=1} \quad \text{or} \quad \mu = \left.\frac{\mathrm{d}\,\mathrm{M}(t)}{\mathrm{d}t}\right|_{t=0}$$

$$\sigma^2 = \left.\frac{\mathrm{d}^2\,\mathrm{G}(t)}{\mathrm{d}^2 t}\right|_{t=1} + \mu - \mu^2 \quad \text{or} \quad \sigma^2 = \left.\frac{\mathrm{d}^2\,\mathrm{M}(t)}{\mathrm{d}^2 t}\right|_{t=0} - \mu^2$$

3.2 Exponential

The following result **must be proved** before its use.

Moment generating function $\mathrm{M}(t) = \mathrm{E}(\mathrm{e}^{tX}) = \dfrac{\lambda}{\lambda - t}$

The following properties may then be used to find the mean and variance.

$$\mu = \left.\frac{\mathrm{d}\,\mathrm{M}(t)}{\mathrm{d}t}\right|_{t=0} \qquad \sigma^2 = \left.\frac{\mathrm{d}^2\,\mathrm{M}(t)}{\mathrm{d}^2 t}\right|_{t=0} - \mu^2$$