AQA GCSE MATHS

HIGHER

Advance Sample Chapter

OXFORD

21 Sequences

Introduction

Musical scales are typically written using eight notes: the C Major scale uses C D E F G A B C. The interval between the first and last C is called an octave.

The pitch of a musical note, measured in Hertz (Hz), corresponds to the number of vibrations per seconds.

The frequencies of the corresponding notes in each octave follow a geometric sequence. If the C in one octave is 130.8 Hz then the C in the next octave is $2 \times 130.8 = 261.6$ Hz, the next C is $2 \times 261.6 = 523.2$ Hz, etc.

What's the point?

Understanding the relationship between terms in a sequence lets you find any term in the sequence and begin to understand its properties.

Objectives

By the end of this chapter you will have learned how to …

- Generate a sequence using a term-to-term or position-to-term rule.
- Recognise a linear sequence and find a formula for its nth term.
- Recognise a quadratic sequence and find a formula for its nth term.
- Recognise and use special types of sequences.

Check in

1 Complete the next two values in each pattern.

    a 2, 4, 6, 8, ...   **b** 100, 94, 88, 82, 76, ...   **c** 1, 2, 4, 7, 11, ...

    d 10, 7, 4, 1, ...   **e** 3, 6, 12, 24, ...   **f** $1, \dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}, \dfrac{1}{5}, \ldots$

2 Given that $n = 3$, put these expressions in ascending order.
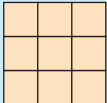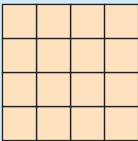
    $2n + 7$ $4(n - 1)$ $2n^2$ $\dfrac{9}{n} + 15$ $15 - n$

3 This pattern has been shown in three different ways.
It has a special name.
What is it called?
Describe how it got this name.

1	4	9	16
1×1	2×2	3×3	4×4

Chapter investigation

Abi, Bo and Cara are making patterns with numbers.

Abi makes a sequence by adding a fixed number onto a starting number.

1, 4, 7, 10, 13, 16, 19, 22, 25, 28, ...   Start with 1, add 3 each time

Bo makes a second sequence by adding Abi's sequence onto another starting number.

1, 2, 6, 13, 23, 36, 52, 71, 93, 118, ...   Start with 1, then add 1, then add 4, then add 7 then add 10, etc.

Cara takes the first two numbers in Bo's sequence and makes a third term by adding these together, then a fourth term by adding the second and third terms, etc.

1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ... $1 + 2 + 3, 2 + 3 = 5, 3 + 5 = 8$, etc.

How do sequences like these behave?

21.1 Linear Sequences

Linear sequences can be generated and described using
- a **'term-to-term' rule**
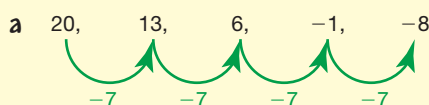- a **'position-to-term'** rule

For the sequence
5, 8, 11, 14, 17…
'add 3'
nth term $= 3n + 2$
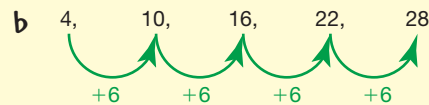
EXAMPLE

Find the first five terms of the following sequences using the given term-to-term rules.

a First term 20   Rule Subtract 7   **b** First term 4   Rule Add 6

a 20,   13,   6,   −1,   −8
-7 -7 -7 -7

b 4,   10,   16,   22,   28
$+6$ $+6$ $+6$ $+6$

EXAMPLE

The nth term of a sequence is $5n - 2$.

a Find the first five terms of the sequence **b** Find the 50th term.

> '5n' means '5 × n'

a 3, 8, 13, 18, 23

1st term ($n = 1$): $(5 \times 1) - 2 = 3$
3rd term ($n = 3$): $(5 \times 3) - 2 = 13$
5th term ($n = 5$): $(5 \times 5) - 2 = 23$

b 50th term = $(5 \times 50) - 2 = 248$

2nd term ($n = 2$): $(5 \times 2) - 2 = 8$
4th term ($n = 4$): $(5 \times 4) - 2 = 17$

EXAMPLE

The rule for a sequence is $T(n) = 2n + 3$, where $T(n)$ is the nth term of the sequence and n is the position of the term in the sequence.

Find the first three terms of the sequence.

> '2n' means '2 × n'

$T(1) = 2 \times 1 + 3 = 5$
$T(2) = 2 \times 2 + 3 = 7$
$T(3) = 2 \times 3 + 3 = 9$
Sequence is 5, 7, 9, …

> T(1) is the first term of the sequence.
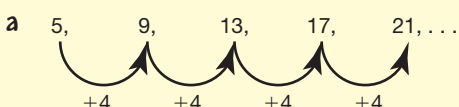> T(2) is the second term of the sequence.

● Linear sequences are sequence in which first differences between terms are a constant.

EXAMPLE

Find the nth term for these sequences.

a 5, 9, 13, 17, 21, …   **b** 10, 4, −2, −8, −14, …

a 5,   9,   13,   17,   21, . . .
$+4$ $+4$ $+4$ $+4$

The difference between each term is '+4'.
nth term = $4n \pm \square$
Compare the sequence with the first terms of the sequence '$4n$'.
$4n$:   4, 8, 12, 16, 20, …   ⬇ +1
        5, 9, 13, 17, 21,
nth term = $4n + 1$

b 10,   4,   −2,   −8,   −14, . . .
-6 -6 -6 -6

The difference between each term is '−6'.
nth term = $-6n \pm \square$
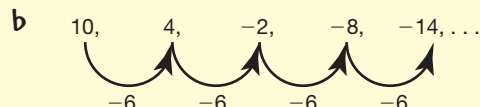Compare the sequence with the first terms of the sequence '$-6n$'.
$-6n$:   −6, −12, −18, −24, …   ⬇ +16
         10,   4,   −2,   −8, …
nth term = $-6n + 16$
          = $16 - 6n$

Exercise 21.1S

1 Find the first four terms of the following sequences using these term-to-term rules.

  a First term 2   Rule Add 6

  b First term 3   Rule Add 5

  c First term 25   Rule Add 3

  d First term −10   Rule Add 3

  e First term: 2.5   Rule Add 5

  f First term $1\frac{3}{4}$ Rule Add $\frac{1}{2}$

  g First term 8   Rule Add 2.5

  h First term 1.2   Rule Add 0.15

2 Find the first four terms of the following sequences using these term-to-term rules.

  a First term 20   Rule Subtract 3

  b First term 102   Rule Subtract 5

  c First term 30   Rule Subtract 7

  d First term 12.5   Rule Subtract 3

  e First term 5   Rule Subtract 4

  f First term −1   Rule Subtract 4

  g First term 9   Rule Subtract 1.5

  h First term $1\frac{3}{4}$ Rule Subtract $\frac{1}{2}$

3 Find the missing terms in these linear sequences.

  a 3, 8, □, 18, 23

  b 4, 7, □, 13, 16

  c 20, □ , 26, □, 32

  d 11, 15, □, 23, □

  e 30, 24, □, 12, 6

  f 10, 7, □, 1, □

  g $2\frac{1}{4}$, □, $1\frac{1}{4}$, □, $\frac{1}{4}$

  h 1.4, 1.25, □, □, 0.8

4 The rule for a sequence is $T(n) = 3n + 3$, where $T(n)$ is the nth term of the sequence and n is the position of the term in the sequence.

  Find the first three terms of the sequence.

5 The rule for a sequence is $T(n) = 5n - 4$, where $T(n)$ is the nth term of the sequence.

  Find the first three terms of the sequence.

6 The terms of a sequence can be generated using the rule $T(n) = 4n + 7$.

  Calculate the

  a 10th term   **b** 15th term

  c 100th term   **d** 1 000th term.

7 Find the first five terms of these sequences.

  a $3n$ **b** $4n + 2$ **c** $2n + 5$

  d $4n - 1$ **e** $5n - 3$ **f** $2n - 6$

  g $3n - 12$ **h** $0.5n + 5$

  i $2.5n - 0.5$ **j** $2 - \frac{1}{3}n$ **k** $\frac{1}{6} + \frac{1}{2}n$

8 Find the first five terms of these sequences.

  a $-2n$ **b** $15 - n$

  c $10 - 2n$ **d** $25 - 5n$

  e $7.5 - 6n$ **f** $4 - 0.5n$

9 Find the nth term of these sequences.

  a 3, 5, 7, 9, 13, …

  b 3, 7, 10, 13, 16, …

  c 5, 8, 11, 14, 17, …

  d 4, 10, 16, 22, 28, …

  e 7, 17, 27, 37, …

  f 2, 10, 18, 26, 34, …

  g 2, 3.5, 5, 6.5, 8, …

  h 1.4, 2, 2.6, 3.2, 3.8, …

10 Find the nth term of the sequences:

  a 15, 12, 9, 6, 3, …

  b 10, 6, 2, −2, −6, …

  c 5, 1, −3, −7, −11, …

  d −2, −5, −8, −11, −14, …

  e $\frac{1}{2}$, $-\frac{1}{4}$, -1, $-1\frac{3}{4}$, $-2\frac{1}{2}$, …

11 For the sequence 2, 6, 10, 14, 18, …

  a predict the 10th term

  b find the nth term.

  c Use your answer to part **b**, to evaluate the accuracy of your prediction.

12 Is 75 a term in the sequence described by the nth term $5n - 3$?

  Give your reasons.

21.1 Linear Sequences

Know how to generate and describe linear sequences using
- the 'term-to-term' rule and
- 'position-to-term' rule using the nth term.

2, 6, 10, 14, 18, ...
differences all = +4
nth term rule = $4n \pm \square$

	2,	6,	10,	14,	18
Compare	4,	8,	12,	16,	20

$-2, -2, -2, -2, -2$
nth term rule = $4n - 2$

HOW TO

To describe a linear sequence using the nth term
1. Find the *constant* difference between terms.
2. This difference is the first part of the nth term.
3. Add or subtract a constant to adjust the expression for the nth term.

EXAMPLE

Ellie says

'there will be 30 squares in the 12th diagram because there are 10 squares in the 4th diagram.'

pattern 1   pattern 2   pattern 3   pattern 4
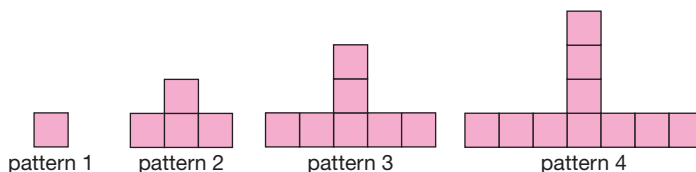
Do you agree with Ellie? Explain your reasoning.

Pattern, n	1	2	3	4
Number of squares, $S(n)$	1	4	7	10

         3   3   3   ① Calculate the first differences.

nth term, $S(n) = 3n \pm \square$ ② First difference is a constant, +3.

③ Compare the sequence with the first term of the sequence $3n$.

$S(n)$ 1,   4,   7,   10, ...
$3n$ 3,   6,   9,   12, ...
      $-2,$ $-2,$ $-2,$ -2

Just because $12 = 3 \times 4$ does not mean $S(12) = 3 \times S(4)$

nth term, $S(n) = 3n - 2$
$S(12) = 3 \times 12 - 2 = 36 - 2 = 34$

No, I do not agree with Ellie. There are 34 squares in the 12th diagram.

EXAMPLE

Jess is generating a sequence using the rule $T(n) = 6n - 2$.
She thinks that every term will be an even number.
Do you agree with Jess? Explain your reasoning.

$T(1) = 4$, $T(2) = 10$, $T(3) = 16$ .... $T(10) = 58$ Test the conjecture.
Yes, I agree with Jess.
When you multiply any number by 6 you produce an even number.
Subtracting two from an even number always produces an even number.

You can also write $T(n) = 2(3n - 1)$ which is clearly a multiple of 2.

Exercise 21.1A

1 Kate thinks that the tenth term of the sequence 3, 7, 11, 15, 19, … will be 38.

Do you agree with Kate?
Give your reasons.

2 Anika says that the nth term of the sequence 5, 9, 13, 17, 21, … is $n + 4$.
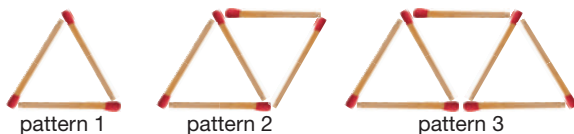
Do you agree with Anika?
Give your reasons.

3 **a** Match each sequence with the correct 'term-to-term' rule and 'position-to-term' rule.

4, 1, −2, −5, …
6, 10, 14, 18, …
3, 10, 17, 24, …
6, 2, −2, −6, …

Subtract 4
Add 4
Subtract 3
Add 7

$T(n) = 4n + 2$
$T(n) = 7n − 4$
$T(n) = 7 − 3n$
$T(n) = 10 − 4n$

b Create your own puzzle

4 Sam is making patterns using matches.

pattern 1   pattern 2   pattern 3

a Find a formula for the number of matchsticks, m, in the nth pattern.

b How many matches will Sam need for the 50th pattern?

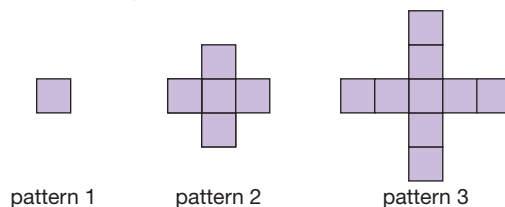**c** Which is the first pattern that will need more than 100 matches to make?

5 Draw a set of repeating patterns to represent the sequence described the nth term $4n + 3$.

6 Matthew is building a fence.

40 cm
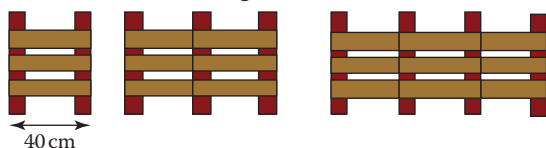
The fence needs to be 11.2 m long. How many posts and planks does Matthew need?

7 How many squares are in the 15th pattern?
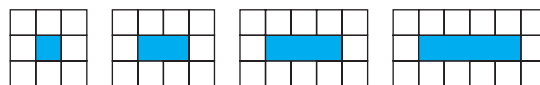

pattern 1   pattern 2   pattern 3

8 Sandra is generating a sequence using the rule $T(n) = 4n − 2$. She thinks that every term will be an even number.

Do you agree with Sandra?

Give your reasons.

9 The first part of this number has been blotted out. Jessica says the number will appear in the sequence $5n − 2$. How can she be sure?

10 Hannah is placing paving slabs around different size ponds.

She notices that the sequence can be described by the nth term $2n + 6$.

a Justify that the nth term is $2n + 6$

b Explain how the expression for the nth term relates to the structure of Hannah's patterns.

11 For each of these linear sequences find the formula for the nth term.

a 5th term 63, decreases by 7 each time.

b 7th term 108, 8th term 113.

c 100th term 26, 101st term 31.

d 22nd term 104, 122nd term 172.

*12 Reya says

'The two sequences $T(n) = 7 + 5n$ and $S(n) = 54 − 6n$ don't have any terms in common because when I solve $T(n) = S(n)$ I get $n = 4\frac{3}{11}$ which isn't an integer'.
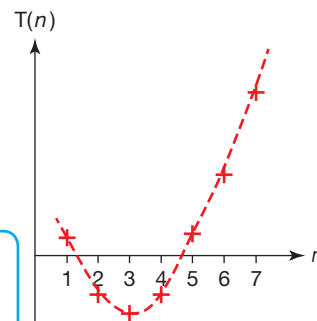
Give reasons why Reya is wrong.

21.2 Quadratic sequences

Quadratic sequences can be generated and described using a
- 'term-to-term' rule
- 'position-to-term' rule.

For the sequence 2, 5, 10, 17, 26...
'add the odd numbers starting with 3.'
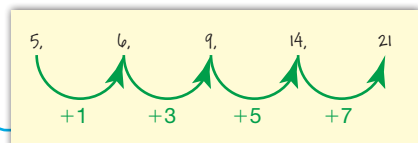nth term = $n^2 + 1$

▲ If you plot T(n) against n for a quadratic sequence then you get points on a quadratic curve.

EXAMPLE

Find the first five terms of the sequence using this term-to-term rule.

First term   5   Rule   Add consecutive odd numbers

5,   6,   9,   14,   21

+1   +3   +5   +7

- Quadratic sequences are sequences in which
  - the second differences between terms are constant
  - the nth term rule contains an n^2 term.

EXAMPLE

The nth term of a sequence is $n^2 + n + 1$.

  a Find the first five terms of the sequence. **b** Check the sequence is a quadratic sequence.

a 3, 7, 13, 21, 31, ...

T(1) = $1^2 + 1 + 1 = 3$
T(2) = $2^2 + 2 + 1 = 7$
T(3) = $3^2 + 3 + 1 = 13$
T(4) = $4^2 + 4 + 1 = 21$
T(5) = $5^2 + 5 + 1 = 31$

T(1) is the first term of the sequence.
T(2) is the second term of the sequence.

b 3, 7, 13, 21, 31, ...

    4   6   8   10   First difference

     2   2   2   Second difference

As the second difference is constant, the sequence is quadratic.

EXAMPLE

Find the nth term for these quadratic sequences.

  a 3, 8, 15, 24, 35, …   **b** 5, 11, 21, 35, 53, …

a 3, 8, 15, 24, 35, ...

    +5 +7 +9 +11 First difference

    +2 +2 +2   Second difference

The coefficient of n^2 is half the value of the second difference.

nth term = $n^2 \pm \square$

Compare the sequence with the first term of the sequence 'n^2'.

        3,   8,   15, 24, 35
$-n^2$ 1,   4,   9, 16, 25
        2,   4,   6,   8, 10

By inspection
Linear sequence, nth term = $2n$
Quadratic sequence, nth term = $n^2 + 2n$

b 5, 11, 21, 35, 53, ...

    +6 +10 +14 +18 First difference

     +4   +4   +4   Second difference

nth term = $2n^2 \pm \square$

Compare the sequence with the first terms of the sequence '$2n^2$'.

        5,   11, 21, 35, 53
$-2n^2$ 2,   8, 18, 32, 50
        3,   3,   3,   3,   3

Constant difference +3
Quadratic sequence, nth term = $2n^2 + 3$

Exercise 21.2S

1 Find the first four terms of these sequences using the term-to-term rules.

   a First term   2
   Rule   Add consecutive odd numbers

   b First term   5
   Rule   Add consecutive even numbers

   c First term   10
   Rule   Add consecutive whole numbers

   d First term   5
   Rule   Add consecutive multiples of 3

   e First term   10
   Rule   Subtract consecutive odd numbers

   f First term   8
   Rule   Subtract consecutive even numbers

2 Find the next two terms of these quadratic sequences.

   a 3, 7, 13, 21, □, □

   b 4, 6, 10, 16, □, □

   c 1, 5, 10, 16, □, □

   d 1, 4, 9, 16, □, □

   e 10, 7, 6, 7, □, □

   f 8, 18, 24, 26, □, □

   g 10, 7, −1, −14, □, □

3 Find the missing terms in these quadratic sequences.

   a 6, 12, 22, □, 54

   b □, 8, 18, 32, 50

   c 2, □, 10, 17, 26

   d 1, 5, □, 19, □

   e −4, 10, 28, □, □

   f 6, 12, 15, □, 12

   g 5, 8, 6, □, −13

4 **a** Find the first five terms for the sequences described by these rules.

   i $T(n) = n^2 + 3$

   ii $T(n) = n^2 + 2n$

   iii $T(n) = n^2 + n - 2$

   iv $T(n) = 2n^2$

   v $T(n) = 2n^2 + 4$

   b Calculate the 10th term of each sequence.

5 **a** Using the nth term for each sequence, calculate the first five terms.

   i $n^2 + 1$ **ii** $n^2 - 3$

   iii $n^2 + 2n + 1$ **iv** $n^2 + n - 4$

   v $2n^2 + n$ **vi** $0.5n^2 + n + 1$

   vii $2n^2 + 2n + 2$ **viii** $n(n - 2)$

   b Calculate the second difference in each case to check the sequences are quadratic.

6 Find the nth term of these quadratic sequences.

   a 4, 7, 12, 19, … **b** −3, 0, 5, 12, …

   c 2, 8, 18, 32, … **d** 2, 6, 12, 20, …

   e 5, 12, 21, 32, … **f** 3, 8, 15, 24, …

   g 6, 14, 26, 42, … ***h** 3, 4, 3, 0, …

7 Find the nth term of these sequences.

   a 9, 6, 1, −6, … **b** 18, 12, 2, −12, …

   c 0, −2, −6, −12, … **d** 10, 8, 4, −2, …

8 **a** For the sequence 2, 6, 12, 20, 30, … predict the position of the first term that will have a value greater than 1000.

   b Find the nth term for the sequence 2, 6, 12, 20, 30, …

   c Use your answer to part **b** to evaluate the accuracy of your prediction.

9 Is 150 a term in the quadratic sequence described by the nth term $n^2 + 3$?

   Give your reasons.

10 Kirsty is generating a sequence using the rule $T(n) = 3n^2 - n$. She thinks that every term will be an even number. Do you agree with Kirsty? Give your reasons.

> **Did you know…**
>
>
>
> The distance travelled by a falling object in successive seconds forms a quadratic sequence.

21.2 Quadratic sequences

Know how to generate and describe quadratic sequences using
- the 'term-to-term' rule and
- the 'position-to-term' rule using the nth term.

To describe a quadratic sequence using the nth term
1. Find the first difference between each term.
2. Find the difference between the first differences: the sequence is quadratic if the second difference is *constant*.
3. The coefficient of n^2 is half the value of the second difference.
4. Add a linear sequence to adjust the expression for the nth term.

$3, 10, 21, 36, 55, \ldots$

second differences all $= +4$

nth term rule $= 2n^2 \pm \square$

$$\begin{array}{cccccc} & 3, & 10, & 21, & 36, & 55 \\ \text{Compare:} & 2, & 8, & 18, & 32, & 50 \\ & 1, & 2, & 3, & 4, & 5 \end{array}$$
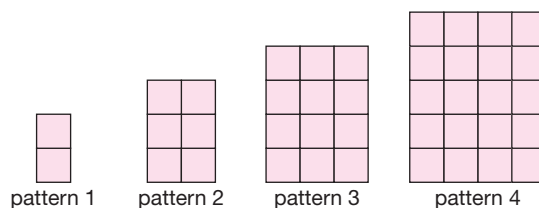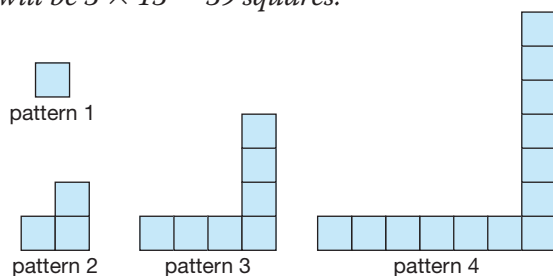
nth term rule $= 2n^2 + n$

Who is right, Ed or Asha? Give your reasons.

Ed says,

'Since $12 = 3 \times 4$ and there are 13 squares in the 4th pattern then in the 12th pattern there will be $3 \times 13 = 39$ squares.'

Asha says,

'There is a pattern in this sequence with 420 squares.'

pattern 1

pattern 2   pattern 3   pattern 4   pattern 1   pattern 2   pattern 3   pattern 4

$$\begin{array}{cccc} 1, & 3, & 7, & 13, \ldots \\ & 2 & 4 & 6 \\ & & 2 & 2 \end{array}$$

① First difference
② Second difference
③ $1 = 2 \div 2$

The coefficient of n^2 is 1.

nth term $= n^2 + \square$

$$\begin{array}{cccc} 1, & 3, & 7, & 13, \ldots \\ 1 & 4 & 9 & 16 \\ \hline 0 & -1 & -2 & -3 \end{array}$$

④ By inspection

Linear sequence, nth term $= 1 - n$

Quadratic sequence, nth term $= n^2 - n + 1$

The 12th pattern has $12^2 - 12 + 1 = 133$ squares.

$$\begin{array}{cccc} 2 & 6 & 12 & 20, \ldots \\ & 4 & 6 & 8 \\ & & 2 & 2 \end{array}$$

The coefficient of n^2 is 1.

nth term $= n^2 + \square$

$$\begin{array}{cccc} 2 & 6 & 12 & 20, \ldots \\ 1 & 4 & 9 & 16 \\ \hline 1 & 2 & 3 & 4 \end{array}$$

Linear sequence, nth term $= n$

Quadratic sequence, nth term $= n^2 + n = n(n + 1)$

$n^2 + n = 420$ Is there a positive integer solution?

$n^2 + n - 420 = (n + 21)(n - 20) = 0$

$n = -21$ or 20

The 20th pattern has 420 squares.

Ed is wrong and Asha is correct.   Remember to answer the question.

Algebra   Sequences

Exercise 21.2A

1   Georgina thinks that the 10th term of the sequence 4, 7, 12, 19, 28, … will be 56. Do you agree with Georgina? Give your reasons.

2   Bob thinks that the nth term of the sequence 5, 7, 11, 17, 25, … will start with '$2n^2$'.

   a   Do you agree with Bob? Give your reasons.

   b   Find the full expression for the nth term of the sequence.

3   **a**   Match each sequence with the correct 'position-to-term' rule.

   b   Find the missing entries.

2, 6, 12, 20, …	$T(n) = n^2 - n$
3, 0, −5, −12, …	
0, 3, 8, 15, …	$T(n) = n^2 + 2$
	$T(n) = n(n + 1)$
3, 6, 11, 18, …	$T(n) = 4 - n^2$

   c   Create your own matching puzzle.

4   The first terms of a quadratic sequence are 3 and 7. Find five different quadratic sequences with this property.

5   Sam is making patterns using matches.

   a   Draw the next pattern.
   b   Find a formula for the number of matchsticks, m, in the nth pattern.
   c   Find the pattern that contains 136 matches.
   d   How many matches will Sam need for the 50th pattern?

6   Draw two sets of patterns to represent the sequence described by the nth term $n^2 + 2$.

7   Draw a set of rectangular patterns to describe sequences with these nth terms.

   a $T(n) = (n + 2)(n + 3)$
   b $T(n) = 2n^2 + 7n + 3$

8   How many squares are there in the 15th pattern?

pattern 1   pattern 2   pattern 3
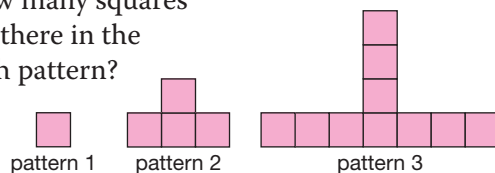
9   Four boys and four girls sit in a row as shown.

B	B	B	B		G	G	G	G

The boys want to swap places with the girls.

G	G	G	G		B	B	B	B

However, they are only allowed to move by either

   – sliding into an empty chair or
   – jumping over one person into an empty chair.

   a   How many moves will it take?
   b   How many moves will it take if there are 3 boys and 3 girls, 2 boys and 2 girls, …?
   c   Copy and complete the table.

Number of pairs	1	2	3	4
Number of moves				

   d   Find a rule to predict the number of moves for 50 pairs of boys and girls.

10   Justify that the nth term of the triangular number sequence 1, 3, 6, 10, … is $\frac{1}{2}n(n + 1)$.

   a   Use an algebraic approach such as analysing the first and second differences to find the nth term.
   b   Use a geometric approach by drawing each term of the sequence.

*11   By looking at successive differences, or otherwise, find expressions for the nth term of these *cubic* sequences.

   a   1, 8, 27, 64, 125, 216, …
   b   2, 16, 54, 128, 250, 432, …
   c   0, 6, 24, 60, 120, 210, …
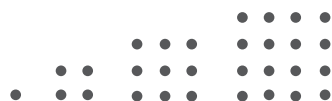   ****d**   −2, 2, 14, 40, 86, 158, …

21.3 Special sequences

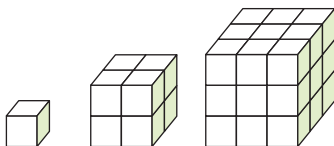● Square, cube and triangular numbers are associated with geometric patterns.

Square numbers

1, 4, 9, 16, 25, ...

Cube numbers

1, 8, 27, 64, 125, ...

Triangular numbers

1, 3, 6, 10, 15, ...

EXAMPLE

a Find a square number that is also **i** a triangular number **ii** a cube number.

b Show that 33, 34 and 35 can each be written as the sum of no more than three triangular numbers.

List the numbers Square numbers   1,   4,   9,   16,   25, (36,) 49, (64,)...

Triangular numbers 1,   3,   6,   10,   15,   21,   28, (36,)...

Cube numbers   1,   8,   27, (64,) 125, 216, ...

a i 1 or 36   **ii** 1 or 64   Look for numbers appearing in both lists.

b 33 = 21 + 6 + 6   34 = 21 + 10 + 3   35 = 28 + 6 + 1

or 15 + 15 + 3   or 28 + 6   or 15 + 10 + 10

● **Arithmetic** (linear) sequences have a constant difference between terms. $T(n + 1) - T(n) = d$

● **Geometric** sequences have a constant ratio between terms. $T(n + 1) \div T(n) = r$

EXAMPLE

Classify these sequences as arithmetic or geometric.

a 5, 8, 11, 14, ...   **b** 1, −2, 4, −8, ...   **c** 3, 3√3, 9, 9√3, ...   **d** 7, 3, −1, −5, ...

a 8 − 5 = 11 − 8 = 14 − 11 = 3   **b** −2 ÷ 1 = 4 ÷ −2 = −8 ÷ 4 = −2

Arithmetic   Constant difference, +3   Geometric   Constant ratio, -2

c 3 √3 ÷ 3 = 9 ÷ 3 √3 = 9√3 ÷ 9 = √3   **d** 3 − 7 = −1 − 3 = −5 − −1 = −4

Geometric   Constant ratio, √3   Arithmetic   Constant difference, -4

● In a **Fibonacci**-type sequence each term is defined as a sum of previous terms.

$T(n + 2) = T(n + 1) + T(n)$ 1, 1, 2, 3, 5, 8, ...   2 = 1 + 1, 3 = 2 + 1, etc.

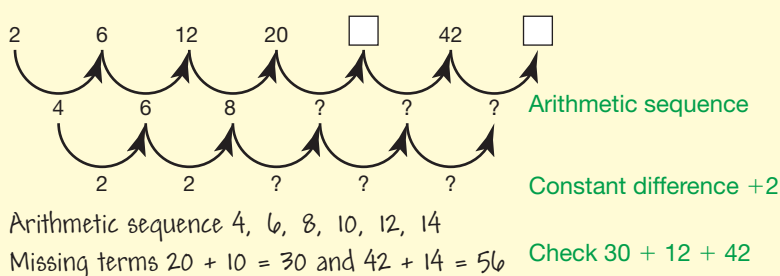$T(n + 2) = 3T(n + 1) - T(n)$ 1, 1, 2, 5, 13, 34, ...   2 = 3 × 1 − 1, 5 = 2 × 3 − 1, etc.

● In a **quadratic** sequence the differences between terms form an arithmetic sequence; the second differences are constant.

EXAMPLE

Find the missing terms in this quadratic sequence.

2, 6, 12, 20, □, 42, □

2   6   12   20   □   42   □

4   6   8   ?   ?   ?   Arithmetic sequence

2   2   ?   ?   ?   Constant difference +2

Arithmetic sequence 4, 6, 8, 10, 12, 14

Missing terms 20 + 10 = 30 and 42 + 14 = 56   Check 30 + 12 + 42

Algebra Sequences

Exercise 21.3S

1 Write down the first ten terms of these number sequences.

  a triangular **b** square **c** cube

2 Express these numbers as the sum of not more than three triangular numbers.

  a 30 **b** 31 **c** 32

3 Describe these sequences using one of these words.

> arithmetic geometric
> quadratic Fibonacci-type

  a 2, 5, 8, 11, ... **b** 7, 11, 15, 19, ...

  c 2, 3, 5, 8, 13, ... **d** 2, 5, 10, 17, ...

  e 2, 6, 18, 54,... **f** 18, 15, 12, 9, ...

  g 1, 4, 5, 9, ... **h** 1, 2, 4, 8, ...

  i 3, 7, 13, 22, ... **j** 0.5, 2, 3.5, 5, ...

  k $\frac{1}{4}, \frac{1}{2}, 1, 2, ...$ **l** $3, 1, \frac{1}{3}, \frac{1}{9}, ...$

  m $-2, 4, -8, 16, ...$ **n** $2, -2, 2, -2, ...$

  o $2, 2 + \sqrt{5}, 2 + 2\sqrt{5}, 2 + 3\sqrt{5}, ...$

  p $\sqrt{5}, 5, 5\sqrt{5}, 25, ...$

4 Do the triangular numbers form a quadratic sequence? Give your reasons.

5 Find the next three terms of the following sequences using the properties of the sequence.

  a Arithmetic $2, 4, \square, \square, \square$

  b Geometric $2, 4, \square, \square, \square$

  c Fibonacci $2, 4, \square, \square, \square$

  d Quadratic $2, 4, \square, \square, \square$

6 Generate the first four terms of a geometric sequence using the following facts.

	First term	Multiplier
a	3	2
b	10	5
c	3	0.5
d	2	-3
e	$\frac{1}{2}$	$\frac{1}{2}$
f	-3	-2
g	4	$\sqrt{3}$
h	$\sqrt{3}$	$\sqrt{3}$
i	$2\sqrt{5}$	$\sqrt{5}$

7 Find the missing term in each sequence, giving your reasons in each case.

  a $5, 10, \square, 20, 25, ...$

  b $5, 10, \square, 40, 80, ...$

  c $5, 10, \square, 25, 40, ...$

  d $5, 10, \square, 26, 37, ...$

8 The geometric mean of two numbers, x and y, is $\sqrt{xy}$. If a, b and c form a geometric sequence, show that b is the geometric mean of a and c.

9 By considering a number's prime decomposition, or otherwise, find three square numbers that are also cube numbers.

10 Generate the next five terms of these Fibonacci-type sequences starting from the first two terms given.

  a $T(n + 2) = T(n + 1) + T(n)$ 2, 1

  b $T(n + 2) = T(n + 1) - T(n)$ 1, 1

  c $T(n + 2) = T(n + 1) + 2T(n)$ 1, 1

  d $T(n + 2) = 2T(n + 1) - T(n)$ 1, 1

  e $T(n + 2) = 2T(n + 1) - T(n)$ 0, 1

11 Ian thinks that the triangular number sequence can be generated using this rule.

$$T(n) = \frac{1}{2}n(n + 1)$$

Do you agree with Ian? Give your reasons.

12 Alice thinks that the sequence 1, 4, 9, 16, 25, ... is a quadratic sequence.

Bob thinks that the sequence 1, 4, 9, 16, 25, ... is a square sequence.

Who is correct? Give your reasons.

13 a Find the first five terms of these sequences.

  i $\dfrac{n}{n + 1}$ **ii** $\dfrac{n}{n^2 + 1}$

  iii $\dfrac{1}{n^2}$ **iv** $\dfrac{n^2}{n + 1}$

  v $(0.9)^n$ **vi** $(1.1)^n$

  b As n increases, comment on the behaviour of $T(n)$.

21.3 Special sequences

- Square numbers 1, 4, 9, 16, … n^2 …
- Cube numbers 1, 8, 27, 64, … n^3 …
- Triangular numbers 1, 3, 6, 10, 15, … – differences 2, 3, 4, 5, …
- Geometric sequences – constant ratio.
- Arithmetic sequence – constant difference.
- Quadratic sequence – differences form an arithmetic sequence.
- Fibonacci-type sequence – each term is the sum of the previous two terms.

HOW TO

① Memorise the square, triangular and cube number sequences.

② Identify arithmetic and quadratic sequences by looking at the difference between terms.

③ Identify geometric sequences by looking at the ratio between terms.

④ Generate Fibonacci-type sequences by adding the previous terms to create the next term.

EXAMPLE

Create two sequences with the following properties.

a Arithmetic sequence with starting term 5

b Geometric sequence with starting term 3

c Fibonacci-type sequence with starting term 4

d Quadratic sequence with starting term 4

a 5, 8, 11, 14, 17, …　　② the rule is 'add 3'

　5, 2, −1, −4, −7, …　　the rule is 'subtract 3'

b 3, 6, 12, 24, 48, …　　③ the rule is '× 2'

　$3\sqrt{5}$, 15, $15\sqrt{5}$, 75, $75\sqrt{5}$, …　　the rule is '× $\sqrt{5}$'

c 4, 5, 9, 14, 23, …　　④ 9 = 5 + 4, 14 = 9 + 5, 23 = 14 + 9

　4, 8, 12, 20, 32, …　　12 = 8 + 4, 20 = 12 + 8, 32 = 20 + 12

d 4, 7, 12, 19, 28, …　　② first differences 3, 5, 7, 9　　second difference is '+2'

　4, 2, −2, −8, −16, …　　first differences −2, −4, −6, −8　　second difference is '−2'

EXAMPLE

Karl loves performing his amazing magic trick.

1 Pick any whole number
2 Write down 4 times the number
3 Add these two numbers together
4 Add the second and third numbers together
5 Add the third and fourth numbers together
6 Keep repeating until you get the 8th number
7 The 8th number is always 60 times the first number

a Try the trick out. Does it always work?

b Does it work for negative integers, fractions and decimals?

c Explain how the trick works.

a Start with 3　　3,　4 × 3 = 12,　3 + 12 = 15,　12 + 15 = 27,　15 + 27 = 42,　27 + 42 = 69,

　　42 + 69 = 111,　69 + 111 = 180　　　　　　　　　　　　Yes　180 = 60 × 3

b Start with −2　　−2,　−8,　−10,　−18,　−28,　−46,　−74,　−120　　Yes −120 = 60 × −2

　Start with $\frac{1}{2}$　　$\frac{1}{2}$,　2,　$2\frac{1}{2}$,　$4\frac{1}{2}$,　7,　$11\frac{1}{2}$,　$18\frac{1}{2}$,　30　　Yes　30 = 60 × $\frac{1}{2}$

　Start with 1.25　　1.25,　5,　6.25,　11.25,　17.5,　28.75,　46.25,　75　　Yes　75 = 60 × 1.25

c Let the first number be N

　N,　4N,　5N,　9N,　14N,　23N,　37N,　60N　　The 8th term is 60 × N

Algebra Sequences

Exercise 21.3A

1 The first term of a sequence is 4. Create five more terms of
  a an arithmetic sequence
  b a geometric sequence
  c a Fibonacci-type sequence
  d a quadratic sequence.

2 Complete the missing cells

	Type	1st term	5th term	Rule
a	Arithmetic	12		$+5$
b	Arithmetic		23	$+4$
c	Arithmetic	0.5		-5
d	Arithmetic			$2n-3$
e	Geometric	3		$\times 4$
f	Geometric		256	$\times 4$
g	Geometric	$\sqrt{3}$		$\times \sqrt{3}$
h	Geometric		$\dfrac{1}{16}$	$\times \dfrac{1}{4}$

3 Jenny thinks that the triangular number sequence can be created by starting with the number 1 and then adding on 2, adding on 3 and so on.
  a Do you agree with Jenny?
    Give your reasons.
  b What are the pentagonal, hexagonal and tetrahedral numbers?
    How are they created?

4 **a** Research and produce a presentation about how Leonardo Fibonacci, also known as Leonardo of Pisa, discovered the Fibonacci sequence 1, 1, 2, 3, 5, …
  b How is the Fibonacci sequence linked to
    i Pascal's Triangle   **ii** the Golden Ratio?

5 Hannah and Sam are given three options for a gift to celebrate their birthdays.
  Option 1 £500
  Option 2 £100 for the first month, £200 the next month, £300 the next month until the end of the year
  Option 3 During the month of their birthday 1p on Day 1, 2p on Day 2, 4p on Day 3, 8p on Day 4, …

5 Hannah's birthday is 20th February. Sam's birthday is 5th September.
  a Which option should Hannah choose? Give your reasons.
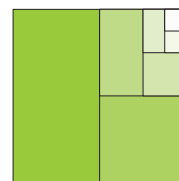 **b** Which option should Sam choose? Give your reasons.

6 **a** Describe this geometric sequence.
    $\dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}, \dfrac{1}{16}, \cdots$
  b A new sequence $S(n)$ is created by adding together the first n terms of the original sequence
    $S(1) = \dfrac{1}{2}$, $S(2) = \dfrac{1}{2} + \dfrac{1}{4} = \dfrac{3}{4}$ etc.
    Write down the first five terms of the sequence $S(n)$.
  c By considering this diagram, or otherwise, explain what $S(n)$ tends to as n gets large.

7 In the second example, how would you change Karl's magic trick so that
  a the 8th term is 99 times the starting number
  b the 12th term is 500 times the starting number?
  c Invent other magic tricks using sequences.

8 Rearrange each set of terms to make a geometric sequence.

  a ab^7 ab^3 ab ab^9 ab^5

  b $c^6 d^3$ $c^2 d^7$ $c^8 d$ $c^4 d^5$ $c^{10} d^{-1}$

  c $24x^4 - 48x^3$ $48x^5 - 96x^6$
     $3x - 6$ $12x^3 - 24x^2$ $6x^2 - 12x$

9 The terms of a geometric sequence are given by $T(n) = ar^{n-1}$ where a and r are constants.
  Describe how the terms $T(n)$ behave for different values of r. Use words like diverge, converge, constant and oscillate.

Summary
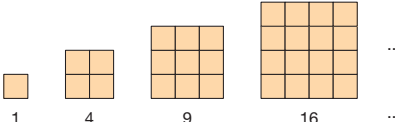
Checkout

You should now be able to...

	Test it Questions
✔ Generate a sequence using a term-to-term or position-to-term rule.	1, 2, 4
✔ Recognise a linear sequence and find a formula for its nth term.	3
✔ Recognise a quadratic sequence and find a formula for its nth term.	4, 5
✔ Recognise and use special sequences.	6 – 9

Language   Meaning   Example

Language	Meaning	Example
Sequence	An ordered set of numbers or other objects.	Square numbers
Term	One of the separate items in a sequence.	
Position	A number that counts where a term appears in a sequence.	1 4 9 16 … $T(1) = 1$ First term, position 1 $T(2) = 4$ Second term, position 2
Term-to-term rule	A rule that links a term in a sequence with the previous term.	Sequence: 3, 5, 7, 9, 11, 13 …. Term-to-term rule: 'add 2' $T(n + 1) = T(n) + 2$
Position-to-term rule / **General term /** **nth term**	A rule that links a term in a sequence with its position in the sequence.	Position-to-term rule: $T(n) = 2n + 1$
Linear / **Arithmetic sequence**	A sequence with a constant difference between terms. A graph of $T(n)$ against n gives points on a straight line.	4, 9, 14, 19, 24, ... Constant difference $= 5$ $T(n + 1) = T(n) + 5$, $T(n) = 5n - 1$
Triangular numbers	The sequence formed by summing the integers: $1, 1 + 2, 1 + 2 + 3, …, \frac{1}{2}n(n + 1), ...$	1, 3, 6, 10, 15, …
Geometric sequence	A sequence with a constant ratio between terms.	1, 3, 9, 27, 81, … Constant ratio $= 3$ $T(n + 1) = 3 \times T(n)$, $T(n) = \frac{1}{3} \times 3^n$
Fibonacci-type sequence	Each term is a sum of previous terms.	1, 1, 2, 3, 5, 8, 13, … $T(n + 2) = T(n + 1) + T(n)$

Quadratic sequence	A sequence in which the differences between terms form an arithmetic sequence; the second differences are constant.
	
	A graph of $T(n)$ against n gives points on a quadratic curve.
		4, 9, 16, 25, …
		  $+5, +7, +9,$ … First difference
		    $2, +2,$ … Second difference
		$T(n + 1) = T(n) + 2n + 3,$
		  $T(n) = n^2 + 2n + 1$

Review

1 a What are the next three terms of these sequences?

    i   8, 17, 26, 35, ...

    ii   71, 58, 45, 32, ...

    iii   2.8, 4.4, 6, 7.6, ...

b Write the term-to-term rule for each of the sequences in part **a**.

2 Calculate the 13th term for the sequences with these position-to-term rules.

  a $T(n) = 5n + 8$

  b $T(n) = 12n - 15$

3 Write a formula for the nth term of these sequences.

  a   1, 7, 13, 19, ...

  b   15, 22, 29, 36, ...

  c   51, 39, 27, 15, ...

  d $-6.5, -8, -9.5, -11, ...$

4 The nth term of a sequence is given by $3n^2 + 4n$.

Calculate

  a   the 7th term

  b   the 10th term.

5 Work out the rule for the nth term of these sequences.

  a   4, 7, 12, 19, ...

  b   2, 10, 24, 44, ...

  c   4, 13, 26, 43, ...

6 Classify each sequence using one of these words.

Linear	Quadratic
Geometric	Fibonacci-type

  a $2, -6, 18, -54, 162, ...$

  b   1, 3, 6, 10, 15, ...

  c   1, 3, 4, 7, 11, ...

  d   0.6, 0.45, 0.3, 0.15, 0, ...

  e   4, 6, 10, 16, 24, ...

  f   0.1, 0.01, 0.001, 0.0001, 0.00001, ...

7 a Write down the next two terms of these sequences.

    i   1, 3, 6, 10, ...   **ii** $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}$

b Write down the nth term of the sequences in part **a**.

8 This sequence is formed by doubling the current term to get the next term.

2, 4, 8, 16, ...

  a   Write down the next three terms of the sequence.

  b   Write down the rule for the nth term of the sequence.

9 Write a rule for the nth term of these sequences and use it to work out the 10th term of each sequence.

  a $\sqrt{2}, 3\sqrt{2}, 5\sqrt{2}\,...$

  b $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, ...$

What next?

Score			
	0 – 3	🟥	Your knowledge of this topic is still developing. To improve look at MyMaths: 1054, 1165, 1166, 1173 and 1920
	4 – 7	🟧	You are gaining a secure knowledge of this topic. To improve look at InvisiPens: 21S01–11
	8	🟩	You have mastered these skills. Well done you are ready to progress! To develop your exam technique look at InvisiPens: 21A01–05

Assessment 21

1   Chris is given the terms of some sequences and the descriptions of the sequences.
He can't remember which description matches which group of terms.
Can you match the each group of terms to the correct description?   [18 marks]

   a   The first five even numbers less than 5.   **i** $-3\ -1\ 1\ 3\ 5$

   b   The first five multiples of 7.   **ii**   5 10 15 20 25 30

   c   The first five factors of 210.   **iii**   11 13 17 19 23 29

   d   The first five triangular numbers.   **iv**   7 14 21 28 35

   f   All the factors of 24.   **v** $4\ 2\ 0\ -2\ -4$

   g   Odd numbers from -3 to 5.   **vi**   1 3 6 10 13 6 10 15

   h   Prime numbers between 10 and 30.   **vii**   1 2 3 4 6 8 12 24

   j $15 <$ square numbers ≤ 81.   **viii**   1 2 3 5 7

   k   Multiples of 5 between 3 and 33.   **ix**   16 25 36 49 64 81

2   **a**   Nathan is given this sequence.

       1,   11,   21,   31,   41,   □,   □

     He says, 'the common difference of this sequence is $+11$'.
     Is he correct? If not, work out the common difference.   [1 mark]

   b   He also says, 'the nth term of this sequence is...'
     Can you complete his sentence? Show your working.   [6 marks]

3   Here is a sequence of patterns.
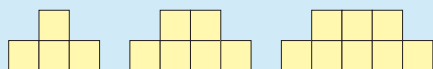
   Find

   a   A pattern in the sequence has an area of $48\,cm^2$. What is its perimeter?

   b   A pattern in the sequence has an perimeter of 48 cm. What is its Area?
     Show your working.   [12 marks]

4   A very famous sequence starts with these numbers.

     1, 1, 2, 3, 5, 8, 13, 21, 34, □, □, □

   a   Work out the term to term rule that shows how the series works.

   b   Using your rule work out the next two terms.

   c   If you are right, the next term after the ones you have just found is a square number.
     What is it?

   d   Do some research to find out what is the name of this famous sequence.
   Can you name it?   [6 marks]

5   Kerry is given some nth terms, T(n), for some sequences. She makes the following
statements. Which of her statements are correct and which are not correct?

   For her statements that are not correct, rewrite the statement correctly.

   a $T(n) = 2n + 7$ The 10th term is 27.

   b $T(n) = 6n - 5$ The first 3 terms are $-5, 1, 7$.

   c $T(n) = 13 - 3n$ The 100th term is 287.

   d $T(n) = n^2 - 10$ The 10th term is 100.

   **e** $T(n) = 15 - 3n^2$ The 100th term is $-29\,985$.   [15 marks]

6 Here is a sequence of patterns.
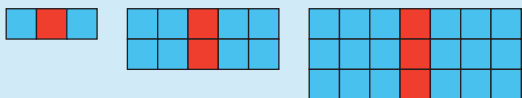
Rachel says that the ratio 'number of red squares' : 'number of blue squares' stays the same as the shapes get bigger. Is she correct? If you think she isn't correct say whether the ratio is decreasing or increasing. Give your reasons. [4 marks]

7 a Jack and Dawn are looking at the start of this sequence of patterns. Jack says the next two patterns will have 12 and then 18 dots. Dawn says that they will have 10 and then 15 dots. Who is correct? Draw the next two of these patterns.
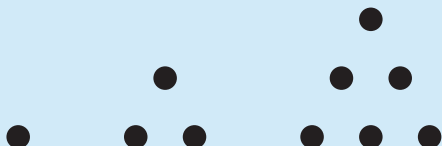
  b Find a formula connecting the pattern number, t, and the number of dots in the pattern, $D(t)$. Show your working and explain your answer.

  c Use your formula to find the number of dots in pattern number

   i 10 **ii** 50 **iii** 100 [12 marks]

8 These diagrams are of a quadrilateral and a pentagon with all their diagonals.

  a How many diagonals are there in the next three polygons?
  Draw the next three of these patterns.
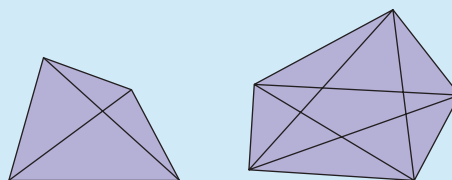
  b Find a formula connecting p, the number of sides in the polygon, and D, the number of diagonals in the pattern of a polygon with 'p' sides.
  Show how you obtain your answer.

  c Use your formula to find the number of diagonals in a polygon with

   i 10 sides **ii** 50 sides **iii** 100 sides. [13 marks]

9 Gareth makes these statements. Which of his statements are true and which are false? Give your reasons. [4 marks]

  a The common difference of the sequence   6,   5,   4,   3,   ...   is 1.

  b The 15th Triangular number is 90.

  c The 20th Triangular number is 175.

  d A sequence can have a common difference of 0.

10 a Fiona writes a linear sequence.
  The third term of a sequence is 4 and the fourth term is 7.
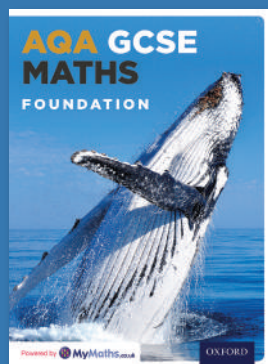 Find the first term.

  b James writes a linear sequence.
  The first term of a sequence is 5 and the tenth term is 25.
  Find the common difference.

  c Robert writes a linear sequence.
  The first term of a sequence is 'a' and the common difference is 'd'.
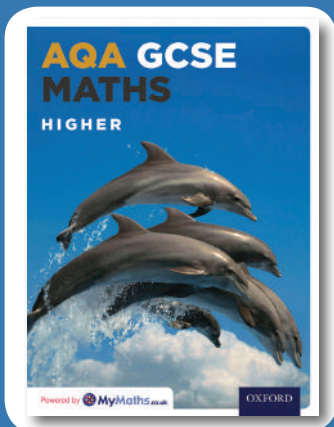 Find, in terms of a and d, the values of the second, third and tenth terms. [9 marks]
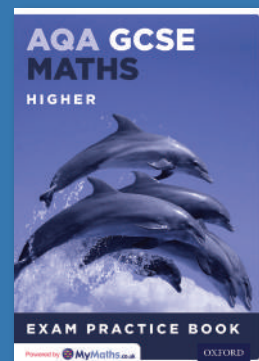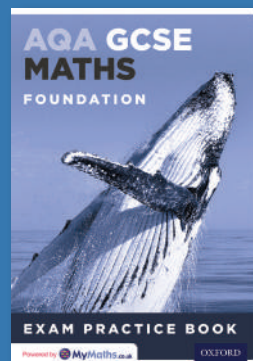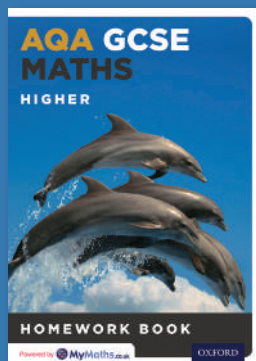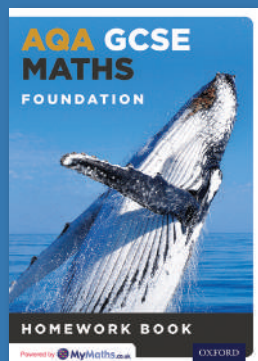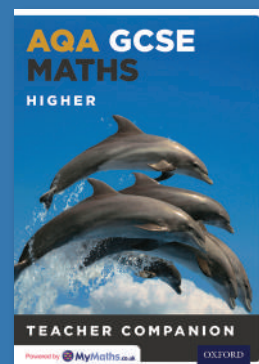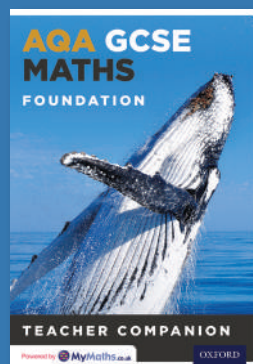
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